Semilinear Calderón Problem on Complex Manifolds

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Joint Work with Leo Tzou

Overview

Historical Account: An Inverse Conductivity Problem

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Consider the conductivity problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u_f) = 0 & \text{in } \Omega \\ u_f = f & \text{on } \partial\Omega \end{cases}$$

where $0 < c \leq \gamma \leq C$ is a conductivity function.

Does the *Dirichlet-to-Neumann (DN) map*

$$\mathcal{N} f = \gamma \partial_\nu u_{|_{\partial\Omega}}$$

defined on suit space of Dirichlet data (depending on γ) determine γ?

A pictorial illustration:

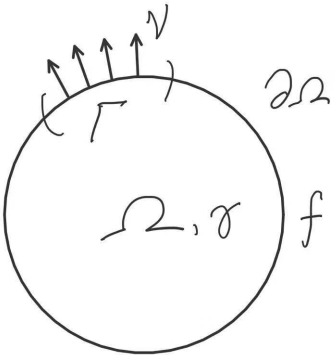

Figure: The Conductivity Problem

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Does the *Dirichlet-to-Neumann (DN) map*

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defined on suit space of Dirichlet data (depending on γ) determine γ?
If Γ is a small open subset of $\partial\Omega$, does the *partial data DN map*

$$\mathcal{N}^\Gamma f = \gamma \partial_\nu u_{|_\Gamma}$$

defined for all f such that $\mathrm{Supp}\, f \subseteq \Gamma$ determine γ?

The conductivity problem can be formulated as an inverse problem for the more well-known *Schrödinger equation*:

$$\begin{cases} (\Delta + V)v_f = 0 & \text{in } \Omega, \\ v_f = f & \text{on } \partial\Omega. \end{cases}$$

where

$$V = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \quad \text{and} \quad \mathcal{N}_V^{\Gamma} f = \partial_\nu v_f|_{\Gamma}.$$

The connection with the inverse problem for conductivity equation is

$$\mathcal{N}_V^{\Gamma} f = \gamma^{-1/2}\mathcal{N}_\gamma^{\Gamma}(\gamma^{-1/2} f) + \frac{1}{2}\gamma^{-1/2}(\partial_\nu\gamma) f \quad \text{on } \Gamma.$$

Question: Let γ_1, γ_2 be two conductivities such that $\mathcal{N}^{\Gamma}_{\gamma_1} = \mathcal{N}^{\Gamma}_{\gamma_2}$ for an arbitrary small open subset $\Gamma \subseteq \partial\Omega$. Is it true that $\gamma_1 = \gamma_2$?

Let

$$V_1 = \frac{\Delta {\gamma_1}^{1/2}}{{\gamma_1}^{1/2}} \quad \text{and} \quad V_2 = \frac{\Delta {\gamma_2}^{1/2}}{{\gamma_2}^{1/2}}.$$

Then $\mathcal{N}^{\Gamma}_{\gamma_1} = \mathcal{N}^{\Gamma}_{\gamma_2}$ implies $\mathcal{N}^{\Gamma}_{V_1} = \mathcal{N}^{\Gamma}_{V_2}$ and $V_1 = V_2$ implies $\gamma_1 = \gamma_2$.

The better question to ask is therefore the following:

Question: Let V_1 and V_2 be two potentials such that $\mathcal{N}^{\Gamma}_{V_1} = \mathcal{N}^{\Gamma}_{V_2}$. Is it true that $V_1 = V_2$?

This can be easily formulated for equations on manifolds, even for nonlinear V.

The Semilinear Calderón Problem

Replacing the potential V by a function $V : \mathbb{C} \times M \to \mathbb{C}$, we get a semilinear equation on a oriented compact, connected Riemannian manifold (M, g) with smooth boundary ∂M:

$$\begin{cases} \Delta_g u_f + V(u_f, x) = 0 & \text{in } M \\ u_f = f & \text{on } \partial M. \end{cases}$$

Assume that V is analytic in the u variable

$$V(u, x) = \sum_{k \geq 2} V_k(x) \frac{u^k}{k!}, \quad V_k(x) = \partial_u^k V(0, x)$$

convergence in the $\mathcal{C}^s(M)$ topology for some $s \in \,]2, \infty[$ not in $\mathbb{N}$.

Proposition (A. Feizmohammadi, L. Oksanen, 2019.)

Let (M, g) be a compact Riemannian manifold with smooth boundary and assume the setting in the previous slide. Then for any f in the set

$$U_\delta = \{f \in \mathcal{C}^s(\partial M) \ / \ \|f\|_{\mathcal{C}^s(\partial M)} < \delta\}$$

there exists a solution u_f to the semilinear equation on the previous slide which is unique in the class

$$V_{C,\delta} = \{u \in \mathcal{C}^s(M) \ / \ \|u\|_{\mathcal{C}^s(M)} < C\delta\}.$$

In particular, u_f is smooth with respect to small perturbations of f.

For $\Gamma \subseteq \partial M$ small the DN maps can now be properly defined as

$$\mathcal{N}^\Gamma : \begin{cases} \mathcal{C}^s(\Gamma) \cap U_\delta \to \mathcal{C}^{s-1}(\Gamma) \\ f \longmapsto \partial_\nu u_{f\,|_\Gamma} \end{cases}$$

Does $\mathcal{N}^\Gamma$ determine V?

Take ϵ_1, ϵ_2 small and set $f_\epsilon = \epsilon_1 f_1 + \epsilon_2 f_2 \in \mathcal{C}^s(\Gamma) \cap U_\delta$. By linearisation

$$\begin{cases} \Delta_g \partial_{\epsilon_j}\big|_{\epsilon_j=0} u_{f_\epsilon} = 0 & \text{in } M, \\ \partial_{\epsilon_j}\big|_{\epsilon_j=0} u_{f_\epsilon} = f_j & \text{on } \partial M. \end{cases}$$

Linearising to the second order, we have

$$\Delta_g \partial_{\epsilon_1} \partial_{\epsilon_2}\big|_{\epsilon_1=\epsilon_2=0} u_{f_\epsilon} = -V_2 (\partial_{\epsilon_1}\big|_{\epsilon_1=0} u_{f_\epsilon})(\partial_{\epsilon_1}\big|_{\epsilon_1=0} u_{f_\epsilon}) = -V_2 v_{f_1} v_{f_2}.$$

Integrating by parts

$$\int_{\partial M} f_3 \, \partial_{\epsilon_1} \partial_{\epsilon_2}\big|_{\epsilon_1=\epsilon_2=0} \mathcal{N}^\Gamma f_\epsilon \, ds_g = \int_{\partial M} f_3 \, \partial_\nu \partial_{\epsilon_1} \partial_{\epsilon_2}\big|_{\epsilon_1=\epsilon_2=0} u_{f_\epsilon} \, ds_g$$
$$= \int_M v_{f_3} \Delta_g \partial_{\epsilon_1} \partial_{\epsilon_2}\big|_{\epsilon_1=\epsilon_2=0} u_{f_\epsilon} \, dv_g = -\int_M V_2 v_{f_1} v_{f_2} v_{f_3} \, dv_g.$$

For two functions V and W such that $\mathcal{N}_V^\Gamma = \mathcal{N}_W^\Gamma$, we have

$$\int_{\partial M} f_3\, \partial_{\epsilon_1} \partial_{\epsilon_2}|_{\epsilon_1=\epsilon_2=0} (\mathcal{N}_V^\Gamma - \mathcal{N}_W^\Gamma)(f_\epsilon)\, ds_g = \int_M (W_2 - V_2) v_{f_1} v_{f_2} v_{f_3}\, dv_g$$

vanishes, where v_{f_j} is harmonic functions with Dirichlet data $f_j \in \mathcal{C}_c^s(\Gamma)$.
Claim:

$$V_2 = W_2 \iff \text{Sufficiently many harmonic functions.}$$

On complex manifolds with Kähler metrics:

$$\text{Holomorphic \& antiholomorphic functions} \implies \text{Harmonic functions}$$

and on Stein manifolds there are a lot of holomorphic functions.

Theorem (Y.Ma, L.Tzou, 2019.)

Let (M, g) be a compact domain of a Stein manifold with smooth boundary and g is a Kähler metric. Assume for two semilinear Schrödinger equations with smooth analytic potentials V and W such that $\mathcal{N}_V = \mathcal{N}_W$, then $V = W$. If (Σ, g) is a Riemann surface, then $\mathcal{N}_V^\Gamma = \mathcal{N}_W^\Gamma$ implies $V = W$ for any open subset $\Gamma \subseteq \partial\Sigma$.

First show that

$$\int_M fuv \, d\mathrm{v}_g = 0$$

for all harmonic functions $u, v \in \mathcal{C}^s(\Gamma)$ implies $f = 0$.

- Boundary Determination via special Dirchlet data.
- Interior determination via stationary phase.

Boundary Determination

Choose locally defined concentrating Dirchlet data with tuned decay

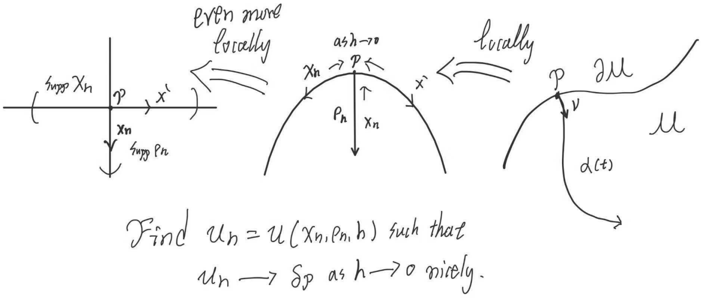

Figure: Boundary Normal Coordinates

In boundary normal coordinates $(x', x_n) = (x', \alpha_\nu(x_n))$.

Lemma (G.Nakamura, K.Tanuma, 2001.)

For every $k \in \mathbb{N}$ there exists polynomials $Q_{j,h}$ in x_n such that

$$v_h = \sum_{0 \le j \le 2k+6} h^{j/2} Q_{j,h} e^{(ix' \cdot \tau - x_n)/h}, \quad \text{and}$$

$$Q_{j,h} = \sum_{0 \le i \le j} h^{-i} q_{ij}(x'/\sqrt{h}) x_n^i \quad \text{for all} \quad j \ge 1, \quad Q_{o,h} = \chi_h,$$

with q_{ij} supported in $(|x'| \le 1)$. Moreover,

$$|\Delta_g v_h| = o(h^{k+1})$$

on $(|x'| \le \sqrt{h}) \times \mathbb{R}^+$ as $h \to 0$.

Our harmonic function will look like

$$u_h = \rho_h v_h + R_h, \quad \text{where} \quad \|R_h\|_{L^2} \leq C\|\Delta_g(\rho_h v_h)\|_{L^2}$$

where ρ_h is concentrated localisation in the normal direction. As $h \to 0$:

- $\rho_h v_h$ concentrate at an arbitrary $p \in \Gamma$.
- R_h decay to high orders by construction.

Thus

$$0 = \int_M f|u_h|^2 \, d\mathrm{v}_g = (\partial_\nu^k f)(p) + o(1)$$

as $h \to 0$ for all k. So f vanishes on Γ up to infinite orders.

Interior Determination

Theorem (C.Guillarmou, M.Salo, L.Tzou, 2018)

Let (M, g) be a compact domain of a Stein manifold with smooth boundary. There exists a dense subset S of M such that for any $p \in S$ and $h > 0$, we can find holomorphic function $\Phi \in \mathcal{C}^k(\overline{M}) \cap \mathcal{O}(M)$ such that $\nabla_g \Phi(p) = 0$ and $Im\, \Phi$ is Morse on M up to the boundary.

Theorem (C.Guillarmou, L.Tzou, 2009)

Let (Σ, g) be a Riemann surface with smooth boundary. There exists a dense subset S of Σ such that for any $p \in S$ and $h > 0$, we can find holomorphic function $\Phi \in \mathcal{C}^k(\Sigma) \cap \mathcal{O}(Int\, \Sigma)$ such that Φ is purely real on Γ^c, $\nabla_g \Phi(p) = 0$ and $Im\, \Phi$ is Morse on Σ up to the boundary. Moreover, we can choose $\alpha \in \mathcal{O}(Int\, \Sigma)$ such that α is purely imaginary on Γ^c, $\alpha(p) = 1$ and $\alpha(p') = 0$ to arbitrarily large order for all other critical points of Φ.

The Case of Stein Domains

Now use holomorphic separability to find holomorphic function $\alpha_j(p_1) = 1$, $\alpha_j(p_k)$ if $k \neq j$ and set $\alpha = \alpha_1 \cdots \alpha_N$ where $\{p_1, ..., p_N\}$ are critical points of $\operatorname{Im} \Phi$. We have harmonic functions

$$u_h = e^{\Phi/h}\alpha \quad \text{and} \quad v_h = \overline{e^{\Phi/h}\alpha}.$$

As $h \to 0$, this yields

$$0 = \frac{C}{h}\int_M f u_h v_h \, dv_g = \frac{C}{h}\int_M f e^{i \operatorname{Im} \Phi} |\alpha|^2 \, dv_g = f(p) + o(1).$$

Since f vanishes on the boundary up to infinite order by boundary determination, we have $f = 0$ on S and therefore on M.

The Case of Riemann Surfaces

Choose holomorphic functions

$$u_h = e^{\Phi/h}\alpha + \overline{e^{\Phi/h}\alpha} \quad \text{and} \quad v_h = e^{-\Phi/h}\alpha + \overline{e^{-\Phi/h}\alpha}.$$

On the boundary

$$u_{h|_{\Gamma^c}} = e^{\mathrm{Re}\,\Phi/h}(\mathrm{Im}\,\alpha - \mathrm{Im}\,\alpha) = 0 \quad \text{and} \quad v_{h|_{\Gamma^c}} = e^{-\mathrm{Re}\,\Phi/h}(\mathrm{Im}\,\alpha - \mathrm{Im}\,\alpha) = 0.$$

Now

$$2\mathrm{Re} \int_\Sigma f e^{2i\mathrm{Im}\,\Phi/h}|\alpha|^2\, d\mathrm{v}_g + \int_\Sigma f(\alpha^2 + \overline{\alpha}^2)\, d\mathrm{v}_g = 0.$$

From here we can show that

$$hCf(p)\mathrm{Re}\, e^{2i\mathrm{Im}\,\Phi(p)/h} + \int_\Sigma f(\alpha^2 + \overline{\alpha}^2)\, d\mathrm{v}_g = o(h),$$

so that

$$\int_\Sigma f(\alpha^2 + \overline{\alpha}^2)\, d\mathrm{v}_g = Cf(p)\mathrm{Re}\, e^{2i\mathrm{Im}\,\Phi(p)/h} = o(1).$$

as $h \to 0$. Choose (h_j) such that $\mathrm{Re}\, e^{2i\mathrm{Im}\,\Phi(p)/h_j} = 1$ and letting $j \to \infty$ we have $f = 0$ on Σ by density.

Proof of the Main Theorem

Proof.

We show that

$$\int_M (V_2 - W_2) v_{f_1} v_{f_2} v_{f_3} \, d\mathrm{v}_g = 0 \implies V_2 = W_2$$

which is trivial for the case of Stein domain. If $n = 2$ then

$$(V_2 - W_2) v_{f_1}(p) = 0 \text{ for every } p \in \Sigma.$$

Thus $(V_2 - W_2)uv = 0$ for all harmonic functions such that $\operatorname{Supp} u$, $\operatorname{Supp} v \subseteq \Gamma$. This reduces to the product case, therefore $V_2 = W_2$.

Proof Continued.

Now by an induction argument, suppose $V_{k-1} = W_{k-1}$ and another linearisation

$$\int_M (V_k - W_k) v_{f_1} \cdots v_{f_k} v_{f_{k+1}} \, d\mathrm{v}_g = 0$$

for harmonic functions $v_{f_1}, \ldots, v_{f_k}, v_{f_{k+1}}$ supported in Γ on the boundary. Apply the determination result repeatedly as before we have $V_k = W_k$ and thus for all k. Since V and W are analytic we must have $V = W$. □

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