The Calderón Problem with Unbounded Potential in Two Dimensions

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An inverse problem for the Schrödinger equation

Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M and dimension $n \geq 2$. Consider the Schrödinger equation

$$\begin{cases} (\Delta_g + V) = 0 & \text{in } M \\ u = f & \text{on } \partial M \end{cases}$$

Does the *Dirichlet-to-Neumann (DN) map*

$$\Lambda_V f \overset{\text{def}}{=} \partial_\nu u_{f}{}_{|_{\partial M}}$$

defined on suitable space of Dirichlet data determine V?

Weak formulation for the Dirichlet-to-Neumann map

Suppose that

$$V \in \begin{cases} L^{n/2}(M) & \text{if } n \geq 3, \\ L^{1+}(M) & \text{if } n = 2, \end{cases}$$

and that 0 is not a Dirichlet eigenvalue of $\Delta_g + V$. We define

$$\Lambda_V : H^{1/2}(\partial M) \to H^{-1/2}(\partial M),$$
$$\langle \Lambda_V f_1, f_2 \rangle_{H^{-1/2}, H^{1/2}} \stackrel{\text{def}}{=} \int_M \langle du_{f_1}, dv_{f_2} \rangle_g + V u_{f_1} v_{f_2} \, dv_g,$$
$$u_{f_1}|_{\partial M} = f_1, \;\; v_{f_2}|_{\partial M} = f_2 \;\text{ and }\; (\Delta_g + V) u_{f_1} = 0.$$

This map is self-adjoint.

The main question

The Calderón problem

Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M and dimension $n \geq 2$. Suppose that V_1, V_2 are functions in $W^{k,p}(M; \mathbb{C})$. Assume that $\Lambda_{V_1} = \Lambda_{V_2}$, then $V_1 = V_2$.

This is the ideal result. Some natural questions arise:

1. Geometric constraint on M?
2. Regularity of V?

Known results in dimensions $n \geq 3$

In the case where $M = \Omega$ is flat

1. $V \in \mathcal{C}^\infty(\Omega)$, Sylvester-Uhlmann (1987);
2. $V \in L^{n/2}(\Omega)$ which are small, Chanillo (1990);

If M is conformal to $\mathbb{R} \times M_0$ and M_0 is simple

1. $V \in \mathcal{C}^\infty(M)$, Ferreira-Kenig-Salo-Uhlmann (2009);
2. $V \in L^{n/2}(M)$, Ferreira-Kenig-Salo (2013).

Known results in dimension $n = 2$

In dimension two the methods are different.

1. $V \in W^{1,p}(\Omega), p > 2$, Bukhgeim (2008);
2. $V \in W^{2,p}(M), p > 2$, Guillarmou-Tzou (2009) if M is a compact Riemann surface with smooth boundary;
3. $V \in L^p(\Omega), p > 2$, Blåsten-Imanuvilov-Yamamoto (2015);
4. $V \in L^p(\Omega), p > 4/3$, Blåsten-Tzou-Wang (2019);
5. $V \in L^p(M), p > 4/3$, Ma (2020) when M is as above.

An integral identity

Assume $\Lambda_{V_1} = \Lambda_{V_2}$. Let $(\Delta_g + V_j)u_j = 0$ then

$$0 = \langle \Lambda_{V_1} u_{1|_{\partial M}}, u_{2|_{\partial M}} \rangle_{H^{-1/2},H^{1/2}} - \langle \Lambda_{V_2} u_{1|_{\partial M}}, u_{2\partial M} \rangle_{H^{-1/2},H^{1/2}}$$
$$= \int_M u_1 V_1 u_2 \, d\mathrm{v}_g - \int_M u_1 V_2 u_2 \, d\mathrm{v}_g = \int_M u_1 (V_1 - V_2) u_2 \, d\mathrm{v}_g.$$

Thus the goal is to find as many solutions as possible.

Complex geometric optic (CGO) solutions

A standard technique is to try constructing solutions of the form

$$u = e^{i\Phi/h}(a + r), \ \lim_{h \to 0} r = 0.$$

For this to be true, we need to solve

$$e^{-i\Phi/h}(\Delta_g + V)e^{i\Phi/h} r = -e^{-i\Phi/h}(\Delta_g + V)e^{i\Phi/h} a,$$

$$\|r\| \lesssim_h \|e^{-i\Phi/h}(\Delta_g + V)e^{i\Phi/h} a\|.$$

This is a semiclassical Carleman estimate with weight Φ. To avoid losing two orders of h on the right hand side, it is favorable if $e^{i\Phi/h}a$ is harmonic. Since $\Delta_g = 2\bar{\partial}^\star\bar{\partial} = 2\partial^\star\partial$, this reduces to finding holomorphic and anti-holomorphic functions.

Method of identification

Let $\Phi = \varphi + i\psi$ be a Holomorphic Morse function with a critical point at p_0. Suppose we find solutions

$$u_1 = e^{i\Phi/h}(a + r) \text{ and } u_2 = \overline{e^{-i\Phi/h}(a + s)}.$$

By the integral identity,

$$0 = \int_M u_1(V_1 - V_2)u_2 \, d\mathrm{v}_g = \int_M e^{2i\varphi/h}(V_1 - V_2)|a|^2 \, d\mathrm{v}_g + \int_M e^{2i\varphi/h}(V_1 - V_2)(a\bar{s} + r\bar{a} + r\bar{s}) \, d\mathrm{v}_g.$$

Let $\{p_j\}_j$ be distinct critical points of φ.
By stationary phase, at least if V_1, V_2 are in $H^2_0(M)$, then

$$0 = \sum_j hc_j e^{2i\varphi(p_j)/h}(V_1 - V_2)(p_j)|a(p_j)|^2 \\ + \int_M e^{2i\varphi/h}(V_1 - V_2)(a\bar{s} + r\bar{a} + r\bar{s})\, d\mathrm{v}_g + o(h).$$

So we need an extra order of decay and a special amplitude a which satisfies

$$a(p_0) = 1 \;\text{ and }\; a(p_j) = 0,\; j \neq 0.$$

If V_1, V_2 are not at least $W^{1,p}(M)$ then a weaker formulation of stationary phase is required.

Remark: In higher dimension we rely on Fourier transform instead of stationary phase, so $r = o(1)$ suffices. It is known from experience that in two dimensions Morse weights are necessary. This is why the two dimensional case is exceptionally difficult.

In reality, we construct $r = r_1 + r_2$ where

$$u = e^{i\Phi/h}(a + r_1 + r_2), \quad \|r_2\| = o(h)$$

and r_1 has special structure which allows us to recover the potentials.

Three difficulties

1. Suitable holomorphic Morse exponential weight/amplitude;
2. Carleman estimates with sufficiently good decay (good remainder term for CGO solutions);
3. A weak method of stationary phase.

Holomorphic Morse weight

On Euclidean space, natural weight is $(z - z_0)^2$. On Riemann surface

Theorem (R.C. Gunning and R. Narasimhan, 1967.)

Every open Riemann Surface admits a holomorphic function Ψ with non-vanishing gradient.

Thus we embed $M \subset \tilde{M}$ where $\tilde{M}$ is a bordered Riemann surface with smooth boundary. Let Ψ be holomorphic with non-vanishing gradient on $\tilde{M}$ and choose $\Phi(p; p_0) = (\Psi(p) - \Psi(p_0))^2$ which is locally $(z - z_0)^2$.

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Carleman estimate for conjugated Laplacian

If $V \in L^\infty(M)$, Hilbert space techniques and Riesz theorem suffices. For unbounded V, we first try to tackle the case $V = 0$. Since

$$e^{-i\Phi/h}\Delta_g e^{i\Phi/h} = 2e^{-i\Phi/h}\partial^\star\partial e^{i\Phi/h} = 2\partial^\star e^{-2i\varphi/h}\partial e^{2i\varphi/h},$$

it suffices to invert $e^{-2i\varphi/h}\partial e^{2i\varphi/h}$. On Euclidean space, the natural right inverse for ∂ is the bounded linear map

$$R : W^{k,p}(\Omega) \to W^{k+1,p}(\Omega) : f \longmapsto \int_\Omega \frac{f(\zeta)}{\overline{z-\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{2\pi i}$$

for all $p \in]1,\infty[$. So in this case we can consider $e^{-2i\varphi/h}Re^{2i\varphi/h}$.

For the case of Riemann surface, there is an analogous construction

Proposition (C. Guillarmou and L. Tzou, 2011.)

Let M be a compact Riemann surface with smooth boundary ∂M, then there exists bounded linear operators

$$T : W^{k,p}(M; T^\star_{1,0}M) \to W^{k+1,p}(M), \text{ and}$$
$$T^\star : W^{k,p}(M) \to W^{k+1,p}(M; T^\star_{1,0}M)$$

for all $p \in]1, \infty[$ such that $\partial T = Id$ and $\partial^\star T^\star = Id$. Moreover, if χ is supported in a small holomorphic chart, then

$$T(\chi f d\bar{z}) = \chi' R \chi f + Kf$$

where $\chi'\chi = \chi$ and K has smooth kernel.

It suffices to consider local estimates. We only consider interior charts.

Proposition (E. Blåsten, L. Tzou and J.-N. Wang, 2019.)

Assume $(p, q, r) \in]4/3, 2[\times]4, \infty[\times]2, \infty[$ *and* $1/2 + 1/q \geq 1/p > 1/2$. *If* $\Omega \subset \mathbb{C}$ *is compact, then we have*

$$\sup_{z_0 \in \Omega} \|e^{-\frac{2i}{h} Re\,(z-z_0)^2} R(e^{\frac{2i}{h} Re\,(z-z_0)^2} f)\|_{L^q(\Omega)} \lesssim h^{1-(\frac{1}{p}-\frac{1}{q})} \|f\|_{W^{1,p}(\Omega)}, \text{ and}$$

$$\sup_{z_0 \in \Omega} \|e^{-\frac{2i}{h} Re\,(z-z_0)^2} R(e^{\frac{2i}{j} Re\,(z-z_0)^2} f)\|_{L^\infty(\Omega)} \lesssim h^{\frac{1}{r}} \|f\|_{W^{1,r}(\Omega)}.$$

Idea of proof.

For $z_0 \in \Omega$ fixed, choose a smooth cut-off χ which is identically 1 for $|z| \geq 2$ and 0 for all $|z| \leq 1$. Set $\chi_h(z) = \chi(h^{-1/2}(z - z_0))$ and split

$$R(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2} f) = R(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2} \chi_h f) + R(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2}(1 - \chi_h) f)$$

Estimate the local part by boundedness of R and area

$$\|R(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2}(1-\chi_h)f)\|_{L^q} \lesssim \|(1-\chi_h)f\|_{L^{\frac{2q}{2+q}}} \lesssim h^{1-(\frac{1}{p}-\frac{1}{q})}\|f\|_{W^{1,p}},$$

$$\|R(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2}(1-\chi_h)f)\|_{L^\infty} \lesssim \|(1-\chi_h)f\|_{L^r} \lesssim h^{\frac{1}{r}}\|f\|_{W^{1,r}}.$$

Idea of proof cont.

Since z is away from z_0 on the support of χ_h, we can integrate by parts

$$
\begin{aligned}
&R((e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2}\chi_h f) \\
&\quad = \frac{ih}{2}\Big(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2}\frac{\chi_h f}{\bar{z}-\bar{z}_0} - R\big(e^{\frac{2i}{h}\mathrm{Re}\,(z-z_0)^2}\partial_{\bar{z}}(\frac{\chi_h f}{\bar{z}-\bar{z}_0})\big)\Big)
\end{aligned}
$$

So we gain one order of decay. Sobolev embedding and Hölder's inequality yields the result. □

So by partition of unity we have bounded linear maps

$$\|e^{-2i\varphi/h}Te^{2i\varphi/h}\|_{W_0^{1,p}(M)\to L^p(M)} \lesssim h^{1-(\frac{1}{p}-\frac{1}{q})}, \text{ and}$$
$$\|e^{-2i\varphi/h}Te^{2i\varphi/h}\|_{W_0^{1,r}(M)\to L^\infty(M)} \lesssim h^{\frac{1}{r}}$$

if $(p,q,r)\in\,]4/3,2[\,\times\,]4,\infty[\,\times\,]2,\infty[$ and $1/2+1/q\geq 1/p>1/2$. Then

$$G_\Phi \overset{\text{def}}{=} 2^{-1}e^{-2i\varphi/h}Te^{2i\varphi/h}T^\star$$

inverts $e^{-i\Phi/h}\Delta_g e^{i\Phi/h}$ on M and satisfies

$$\|G_\Phi\|_{L^p(M)\to L^q(M)} \lesssim h^{1-(\frac{1}{p}-\frac{1}{q})}, \quad \|G_\Phi\|_{L^p(M)\to L^\infty(M)} \lesssim h^{0+}.$$

The second estimate follows from interpolation.

Perturbation by V

Since

$$e^{-i\Phi/h}(\Delta_g + V)e^{i\Phi/h}G_\Phi = \mathrm{Id} + VG_\Phi,$$

$$\|VG_\Phi\|_{L^p(M)\to L^p(M)} \lesssim \|V\|_{L^p(M)}\|G_\Phi\|_{L^p(M)\to L^\infty(M)} = o(1).$$

Standard functional analysis shows that as $h \to 0$

$$e^{-i\Phi/h}(\Delta_g + V)e^{i\Phi/h}G_\Phi(\mathrm{Id} + VG_\Phi)^{-1} = \mathrm{Id}$$

is a reasonable formula for the inverse.

Structure of the remainder

Recall that we are trying to solve

$$e^{-i\Phi/h}(\Delta_g + V)e^{i\Phi/h} r = -Va.$$

By the inverse we constructed, we look for

$$r = -G_\Phi(\mathrm{Id} + VG_\Phi)^{-1} Va = -\sum_{j\geq 0} (-1)^j G_\Phi (VG_\Phi)^j Va.$$

Split $r = r_1 + r_2$ for

$$r_1 \stackrel{\text{def}}{=} -\sum_{0\leq j<2} (-1)^j G_\Phi (VG_\Phi)^j Va, \quad r_2 \stackrel{\text{def}}{=} -\sum_{j\geq 2} (-1)^j G_\Phi (VG_\Phi)^j Va.$$

Estimate for r_2

If we define

$$\frac{1}{p'} \overset{\text{def}}{=} \frac{1}{p} - \frac{1}{2}, \quad \frac{1}{q} \overset{\text{def}}{=} \frac{1}{p} + \frac{1}{p'}, \quad \frac{1}{r} \overset{\text{def}}{=} \frac{1}{p} - \frac{1}{4},$$

then $1/2 + 1/p' \geq 1/p > 1/2$ and $1/2 + 1/r \geq 1/q > 1/2$, so using the Carleman estimates for G_Φ, we have

$$\begin{aligned}
\|G_\Phi(VG_\Phi)^j Va\|_{L^r(M)} &\lesssim h^{1-(\frac{1}{q}-\frac{1}{r})}\|(VG_\Phi)^j Va\|_{L^q(M)} \\
&\lesssim h^{1-(\frac{1}{q}-\frac{1}{r})}\|V\|_{L^p(M)}\|G_\Phi(VG_\Phi)^{j-1}Va\|_{L^{p'}(M)}, \\
\|G_\Phi(VG_\Phi)^j Va\|_{L^{p'}(M)} &\lesssim h^{1-(\frac{1}{p}-\frac{1}{p'})}\|(VG_\Phi)^j Va\|_{L^p(M)} \\
&\lesssim h^{1-(\frac{1}{p}-\frac{1}{p'})}\|V\|_{L^p(M)}\|G_\Phi(VG_\Phi)^{j-1}Va\|_{L^\infty(M)}.
\end{aligned}$$

Since $2 < r$ and $4 < p'$, as well as $1 - (\frac{1}{q} - \frac{1}{r}), 1 - (\frac{1}{p} - \frac{1}{p'}) > \frac{1}{2}$, we have

$$\|G_\Phi(VG_\Phi)^j Va\|_{L^4(M)} \lesssim h^{\frac{1}{2}+}\|V\|_{L^p(M)}\|G_\Phi(VG_\Phi)^{j-1}Va\|_{L^\infty(M)},$$

$$\|G_\Phi(VG_\Phi)^j Va\|_{L^2(M)} \lesssim h^{1+}\|V\|_{L^p(M)}\|G_\Phi(VG_\Phi)^{j-2}Va\|_{L^\infty(M)}.$$

Moreover

$$\|G_\Phi(VG_\Phi)^j Va\|_{L^\infty(M)} \lesssim h^{0+}\|V\|_{L^p(M)}\|G_\Phi(VG_\Phi)^{j-1}Va\|_{L^\infty(M)}.$$

Thus

$$\sup_{p\in\Omega}\|r_2\|_{L^2(M)} = o(h) \quad \text{and} \quad \sup_{p\in\Omega}\|r_2\|_{L^4(M)} = o(h^{\frac{1}{2}}).$$

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Recovery of the potential

Now we know how to construct solutions

$$u_1 = e^{i\Phi/h}(a + r) \text{ and } u_2 = \overline{e^{-i\Phi/h}(a + s)}.$$

Assume for the moment that we have proved

$$\int_M e^{2i\varphi/h}(V_1 - V_2)|a|^2 \, d\mathrm{v}_g = o_{L^2}(h).$$

Can we show that $V_1 = V_2$?

Weak method of stationary phase

A standard way is

Lemma (E. Blåsten, O.Y. Imanuvilov and M. Yamamoto, 2015.)

For every $f \in L^2(\mathbb{R}^2)$ we have

$$\lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}^2} e^{\frac{2i}{h} Re\,(z-z_0)^2} f(z)\, dz d\bar{z} = f(z_0)$$

in $L^2(\mathbb{R}^2)$.

Since our weight is locally $(z - z_0)^2$, we can use this technique.

Identification of the potentials

Proposition (Ma, 2020.)

Let M be a compact Riemann surface with smooth boundary and V_1, V_2 be two potentials in $L^p(M)$ for $p > 4/3$. Assume that $\Lambda_{V_1} = \Lambda_{V_2}$, then $V_1 - V_2 \in L^2(M)$.

Assuming this, note p_0 is a critical point of $\Phi(p;p_0) = (\Psi(p) - \Psi(p_0))^2$ if and only if $\Psi(p) = \Psi(p_0)$. We can construct a holomorphic amplitude

$$a(p;p_0) \overset{\text{def}}{=} \chi(p) - \bar{T}_p\Big(\frac{\bar\partial\chi}{\Psi(p) - \Psi(p_0)}\Big)(\Psi(p) - \Psi(p_0))$$

where χ is cut off and identically 1 near a prescribed $\tilde{p}_0$.

Let $\{p_0, p_1, ..., p_n\}$ be the set of points for which $\Psi(p_j) = \Psi(p_0)$, then we have

$$a(p_j; p_0) = \begin{cases} 1 & \text{if} \quad j = 0, \\ 0 & \text{if} \quad j \neq 0, \end{cases}$$

so that by assumption

$$0 = \lim_{h \to 0} o_{L^2}(1) = \lim_{h \to 0} \frac{1}{h} \int_M e^{2i\varphi/h} (V_1 - V_2) |a|^2 \, d\mathrm{v}_g = V_1(p_0) - V_2(p_0)$$

for almost all p_0 near an arbitrarily prescribed $\tilde{p}_0$.

By compactness $V_1 = V_2$ almost everywhere.

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The final result

Now we prove our main result

Theorem (Y. Ma, 2020.)

Let M be a compact Riemann surface with smooth boundary and V_1, V_2 be two potentials in $L^p(M)$ for $p > 4/3$. Assume that $\Lambda_{V_1} = \Lambda_{V_2}$, then $V_1 = V_2$.

Idea of proof.

We now choose solutions

$$u_1 = e^{i\Phi/h}(a + r_1 + r_2) \quad \text{and} \quad u_2 = \overline{e^{-i\Phi/h}(a + s_1 + s_2)}$$

and implement them into the integral identity

$$\begin{aligned}
&\int_M e^{2i\varphi/h}(V_1 - V_2)|a|^2 \, d\mathrm{v}_g \\
&\quad = \int_M e^{2i\varphi/h}(V_1 - V_2)(a\bar{s}_1 + r_1\bar{a} + r_1\bar{s}_1 + r_1\bar{s}_2 + r_2\bar{s}_1) \, d\mathrm{v}_g \\
&\qquad\qquad + \int_M e^{2i\varphi/h}(V_1 - V_2)(a\bar{s}_2 + \bar{a}r_2 + r_2\bar{s}_2) \, d\mathrm{v}_g .
\end{aligned}$$

Idea of proof cont.

The later terms satisfies

$$\sup_{p_0\in\Omega}\left|\int_M e^{2i\varphi/h}(V_1-V_2)(a\bar{s}_2+\bar{a}r_2+r_2\bar{s}_2)\,dv_g\right|$$

$$\leq \sup_{p\in\Omega}\|V_1-V_2\|_{L^2(M)}\|a\bar{s}_2+\bar{a}r_2+r_2\bar{s}_2\|_{L^2}$$

$$\leq \sup_{p\in\Omega}\|V_1-V_2\|_{L^2(M)}\big(\|a\|_{L^\infty(M)}\|\bar{s}_2\|_{L^2(M)}$$

$$+\|a\|_{L^\infty(M)}\|r_2\|_{L^2(M)}+\|r_2\|_{L^4(M)}\|s_2\|_{L^4(M)}\big)=o(h).$$

Idea of proof cont.

Now we just need that

$$\frac{1}{h}\int_M e^{2i\varphi/h}(V_1 - V_2)(a\bar{s}_1 + r_1\bar{a} + r_1\bar{s}_1 + r_1\bar{s}_2 + r_2\bar{s}_1)\, d\mathrm{v}_g \to 0$$

as $h \to 0$. We note that most of the terms can be estimated by similar strategies as before except for

$$\int_M e^{2i\varphi/h}(V_1 - V_2)aG_{\bar{\Phi}}V_2\bar{a}\, d\mathrm{v}_g \quad \text{and} \quad \int_M e^{2i\varphi/h}(V_1 - V_2)\bar{a}G_{\Phi}V_1 a\, d\mathrm{v}_g$$

Idea of proof cont.

Assume for the sake of clarity that we are on Ω, then

$$\Phi = (z - z_0)^2, \quad G_\Phi = 4^{-1} R_\varphi \bar{R}, \quad R_\varphi = e^{-2i\varphi/h} R e^{2i\varphi/h}, \quad a = 1.$$

Thus

$$\begin{aligned}
&\int_\Omega e^{2i\varphi/h}(V_1 - V_2) G_\Phi V_2 \, dz d\bar{z} \\
&\quad = 4^{-1} \int_\Omega e^{2i\varphi/h}(V_1 - V_2) e^{-2i\varphi/h} R e^{2i\varphi/h} \bar{R} V_2 \, dz d\bar{z} \\
&\quad = -4^{-1} \int_\Omega e^{2i\varphi/h} [\bar{R}(V_1 - V_2)][\bar{R} V_2] \, dz d\bar{z} \\
&\quad = -4^{-1} h [\bar{R}(V_1 - V_2)][\bar{R} V_2] + o_{L^2}(h).
\end{aligned}$$

Idea of proof cont.

Neat trick by Imanuvilov-Yamamoto is to modify the first term in r_1

$$e^{-2i\varphi/h}Re^{2i\varphi/h}\bar{R}V_2 \longmapsto e^{-2i\varphi/h}Re^{2i\varphi/h}(\bar{R}V_2 - \bar{R}(V_1 - V_2)(z_0)).$$

Then in the stationary phase we get

$$\begin{aligned} & -4^{-1}h[\bar{R}(V_1 - V_2)][\bar{R}V_2] + o_{L^2}(h) \\ & \longmapsto -4^{-1}h[\bar{R}(V_1 - V_2)][\bar{R}V_2] + 4^{-1}h[\bar{R}(V_1 - V_2)][\bar{R}V_2] + o_{L^2}(h) \\ & \qquad = o_{L^2}(h) \end{aligned}$$

□

So we can arrange to get

$$\frac{1}{h}\int_M e^{2i\varphi/h}(V_1 - V_2)(a\bar{s}_1 + r_1\bar{a} + r_1\bar{s}_1 + r_1\bar{s}_2 + r_2\bar{s}_1)\, d\mathrm{v}_g = o_{L^2}(1).$$

Letting $h \to 0$ proves the result.

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Recovery of singularities (if time permits)

We still haven't shown that $\Lambda_{V_1} = \Lambda_{V_2} \;\Rightarrow\; V_1 - V_2 \in L^2(M)$.

Idea of proof.

Choose solutions with new holomorphic weights

$$u_1 = e^{-i\bar{\Psi}/2}(a+r) \;\text{ and }\; u_2 = \overline{e^{i\Psi\zeta/2}(a+s)}.$$

Since Ψ has non-vanishing gradient, get better Carleman estimates

$$\|G_\Psi\|_{L^p(M)\to L^q(M)} \lesssim \frac{1}{|\zeta|}, \quad \|G_\Phi\|_{L^p(M)\to L^\infty(M)} \lesssim \frac{1}{|\zeta|^{0+}}$$

for all $1/p + 1/q = 1$.

Idea of proof cont.

Let $\tilde{\rho}$ be a cut off in $\mathbb{C}$ identically 1 on $|\zeta| > 2\epsilon$ and 0 on $|\zeta| < \epsilon$. By integral identity we get

$$\tilde{\rho}(\zeta) \int_M e^{-i\Psi\cdot\zeta} |a|^2 V \, dv_g = \tilde{\rho}(\zeta) o\left(\frac{1}{\zeta}\right)$$

Taking the inverse Fourier transform on both sides, locally the left hand side is Fourier inversion

$$\sum_j \mathcal{F}^{-1} \tilde{\rho} \mathcal{F}(\chi_j V |a|^2 |g_j|^{1/2})$$
$$= \sum_j \mathcal{F}^{-1} \mathcal{F}(\chi_j V |a|^2 |g_j|^{1/2}) - \sum_j \mathcal{F}^{-1} (1 - \tilde{\rho}) \mathcal{F}(\chi_j V |a|^2 |g_j|^{1/2}) \in L^2$$

Idea of proof cont.

Now we have

$$\chi_0 V|g_0|^{1/2} = \sum_j \mathcal{F}^{-1}\mathcal{F}(\chi_j V|a|^2|g_j|^{1/2}) = \mathcal{C}^\infty + L^2.$$

By restriction and compactness we thus get $V \in L^2(M)$. □

Remark on existing techniques

Limitation to $V \in L^p(M)$ for $p > 4/3$ seems required.

Carleman estimate derived by factoring $\Delta_g = \partial^\star \partial = \bar{\partial}^\star \bar{\partial}$ and considering $\bar{\partial}^\star$ and $\bar{\partial}$ separately.

Since ∂ is first order operator, $p > 4/3$ is the natural assumption.

But Δ_g is a second order operator, so we in fact expect $p > 1$.

New methods required for Δ_g to obtain better Carleman estimate!

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