THE CALDERÓN PROBLEM

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ABSTRACT. In this article we provide short mathematical descriptions for the main aspects of the Calderón problem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$, $n \geq 2$. For convenience we assume that $\partial\Omega$ is smooth. Suppose that $\gamma \in L^\infty(\Omega)$ is such that $\gamma > c > 0$ for some $c > 0$, then the following boundary value problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u_f \in H^1(\Omega)$ for every $f \in H^{1/2}(\partial\Omega)$. Thus, the *Dirichlet-Neumann* map Λ_γ can defined weakly on $H^{1/2}(\partial\Omega)$ by

$$\Lambda_\gamma : \begin{cases} H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \\ f \mapsto \gamma \partial_\nu u_{f|_{\partial\Omega}}. \end{cases}$$

Physically, if Ω models an electrical conductor with conductivity γ at each point $x \in \Omega$, then the Dirichlet-Neumann map corresponds to a measurement of the electrical current for each boundary voltage f. In this context, the Dirichlet-Neumann map is also referred to as the voltage-current map. The question Calderón raised was the following: is it possible to recover the electrical conductivity of an object by making infinitely many voltage to current measurements? Mathematically, we wish to prove

Theorem 1. *Let $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ be such that $\gamma_1, \gamma_2 > c > 0$ for some $c > 0$. Suppose that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then we have $\gamma_1 = \gamma_2$.*

Theorem 1 was first proved by Sylvester-Uhlmann in [3] under the slightly stronger condition that $\gamma_j \in \mathcal{C}^2(\overline{\Omega})$ and $\gamma_{1|_{\partial\Omega}} = \gamma_{2|_{\partial\Omega}}$. It is not hard to sketch the proof of their result.

Sketch of the Sylvester-Uhlmann proof. It was observed that if one introduces the Schrödinger potentials

$$q_j \stackrel{\text{def}}{=} \frac{\Delta {\gamma_j}^{1/2}}{{\gamma_j}^{1/2}}, \quad j = 1, 2,$$

then the assumption that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and $\gamma_{1|_{\partial\Omega}} = \gamma_{2|_{\partial\Omega}}$ implies

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0 \tag{1}$$

for all $u_j \in H^1(\Omega)$ solving $(\Delta + q_j)u_j = 0$. For every $\xi \in \mathbb{R}^n$, the crucial contribution by Sylvester-Uhlmann was their constructions of complex geometric optic (CGO) solutions of the form $u_j = e^{x\cdot\zeta_j/h}(1 + r_j)$, $h > 0$ using the L^2 Carleman estimate. Here

$$\zeta_j \in \mathbb{C}^n, \quad \zeta_j \cdot \zeta_j = 0, \quad \zeta_1 + \zeta_2 = ih\xi, \quad (\Delta + q_j)u_j = 0, \quad r_j = o_{L^2}(1).$$

as $h \to 0$. Implementing these solutions into identity (1), we get that

$$\int_\Omega (q_1 - q_2) e^{ix\cdot\xi} \, dx = o(1).$$

Hence letting $h \to 0$ we conclude that the Fourier transform of $q_1 - q_2$ is zero for every $\xi \in \mathbb{R}^n$ so that $q_1 = q_2$. In particular, $\gamma_1 = \gamma_2$. □

In the above we used the linear weight $x \cdot \zeta$ in the Carleman estimate. The case $n = 2$ is considerably different since one relies on holomorphic weight instead. The result in this case is due to Bukhgeim [1].

In recent years it has been a main motivation in the field to study the Calderón problem in the case where the underlying geometry Ω is replaced by a compact, Riemannian manifold (M, g) with smooth boundary ∂M. In order for the method of CGO solutions to apply one needs prove suitable Carleman estimate with weights that depends on the geometry of (M, g). The case where M is a Riemann surface is completed solved [4], but in the higher dimension setting the problem remains open except for some very special cases [2].

Other than uniqueness, one could also study the questions of stability, reconstruction, partial data, or the optimal regularity condition on γ, etc.

References

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