# SEMILINEAR CALDERÓN PROBLEM ON STEIN MANIFOLDS WITH KÄHLER METRIC

YILIN MA AND LEO TZOU

Abstract. We extend existing methods which treat the semilinear Calderón problem on bounded domain to a class of complex manifolds with Kähler metric. Given two semilinear Schördinger operators with the same Dirchlet-to-Neumann data, we show that the integral identities that appear naturally in the determination of the analytic potentials are enough to deduce uniqueness on the boundary up to infinite order. By exploiting the complex structure under assumption, this information allows us to apply the method of stationary phase and recover the potentials in the interior as well.

## 1. Introduction

In this paper, we study the semilinear elliptic equation

$$(NCP_{f,V}) \qquad \begin{cases} \Delta_g u + V(p,u) = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

on a $n$-complex dimensional compact, connected Kähler manifold $(M,g)$ with smooth boundary $\partial M$. Here $V : M \times \mathbb{C} \to \mathbb{C}$ is a $\mathcal{C}^\infty(M)$ function for every complex variable such that

$$V(p,u) = \sum_{k\geq 1} \frac{V_k}{k!} u^k, \quad \text{and} \quad V_k(p) \overset{\text{def}}{=} \partial_u^k V(p,0) \tag{1}$$

converges in the $\mathcal{C}^s(M)$ topology for non-integer $s > 2$. We assume zero is not an eigenvalue for the operator $\Delta_g + V_1$. Moreover

$$\Delta_g u \overset{\text{def}}{=} -\frac{1}{\sqrt{\det g}} \partial_i \big(g^{ij} \sqrt{\det g}\, \partial_j u\big)$$

defined locally is the positive Laplace-Beltrami operator. We assume in addition that $M$ is holomorphic separable and has local charts given by holomorphic functions, in the sense that

- For any $p \neq q$ in $M$, there exists $f \in \mathcal{C}^\infty(M) \cap \mathcal{O}(\operatorname{Int} M)$ such that $f(p) \neq f(q)$.
- For any $p \in M$, there exists $f_1, ..., f_n \in \mathcal{C}^\infty(M) \cap \mathcal{O}(\operatorname{Int} M)$ which forms a complex coordinate system centred at $p$.

In particular, the above covers the case of Stein manifolds. It was shown in [5] that there exists $\delta, r, C > 0$ depending on $(M,g)$ such that if we consider the sets

$$U_\delta \overset{\text{def}}{=} \big\{h \in \mathcal{C}^s(\partial M) \ / \ \|h\|_{\mathcal{C}^s(\partial M)} \leq \delta\big\} \quad \text{and} \quad V_r \overset{\text{def}}{=} \big\{w \in \mathcal{C}^s(M) \ / \ \|w\|_{\mathcal{C}^s(M)} \leq r\big\},$$

then for any $f \in U_\delta$ there exists a unique $u_f \in V_r$ which solves $(NCP_{f,V})$, with the estimate

$$\|u\|_{\mathcal{C}^s(M)} \leq C\|f\|_{\mathcal{C}^s}(M). \tag{2}$$

In particular, $u_f$ is analytic with respect to small complex perturbation of $f$: If $f_\epsilon = \epsilon_1 f_1 + \cdots + \epsilon_k f_k$, $\epsilon = (\epsilon_1, ..., \epsilon_k) \in \mathbb{C}^k$ is complex parameter and $f_1, ..., f_k$ are in $\mathcal{C}^s(M)$, then the solution $u_{f_\epsilon}$ admits a power series representation with respect to the parameter $\epsilon$ in the $\mathcal{C}^k$ topology.

Thus for sufficiently small boundary data, we can define the Dirchlet-to-Neumann map

$$\mathcal{N} : U_\delta \to \mathcal{C}^{s-1}(\partial M) : f \longmapsto \partial_\nu u_{f_{|\partial M}}.$$

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We will prove the following theorem:

**Theorem 1.** *Let $(M, g)$ be specified as above. Let $\mathcal{N}_V$ and $\mathcal{N}_W$ be the Dirchlet-to-Neumann maps corresponding to $V$ and $W$, where $V$ and $W$ are smooth and satisfy condition (1). Assume that $\mathcal{N}_V = \mathcal{N}_W$ and $V_1 = W_1 = 0$, then $V = W$.*

The case where only partial data is available is much harder. Let $\Gamma$ be an arbitrarily small open subset of $\partial M$ and let $\Gamma^c$ denote the complement of $\Gamma$ in $\partial M$. We consider the Dirchlet-to-Neumann map with partial data

$$\mathcal{N}^\Gamma : \left\{h \in U_\delta \ / \ \text{Supp } h \subseteq \Gamma\right\} \to \mathcal{C}^{s-1}(\partial M) : f \longrightarrow \partial_\nu u_{f_{|\Gamma}}.$$

Using the tools developed in [10], we will extend Theorem 1 to solve the semilinear Calderón problem on Riemann surface where only partial data is available. We will show that

**Theorem 2.** *Let $(\Sigma, g)$ be a Riemann surface. Let $\mathcal{N}_V^\Gamma$ and $\mathcal{N}_W^\Gamma$ be the Dirchlet-to-Neumann map with partial data corresponding to $V$ and $W$, where $V$ and $W$ are smooth and satisfy condition (1). Assume that $\mathcal{N}_V^\Gamma = \mathcal{N}_W^\Gamma$, then $V = W$.*

Since every Riemann surface is also Stein, Theorem 2 in particular covers the situation of Theorem 1.

Our strategy will be as follows. In section 2 we formulate our main theorems in terms of a class of integral identities analogous to the linearised Calderón problem [7, 9, 11, 23]. Starting from section 3 our method will begin to differ from [11]. We prove directly a boundary determination result assuming nothing but the integral identity in section 2. This will be done using special solutions to the Laplace equation constructed in [22] via the WKB method. The standard techniques in proving such results are usually based on the theory of pseudodifferential operators. Then in section 4, assuming sufficient regularity on the potential, we can simplify the proofs in [10] and obtain fairly easily interior recovery based on the boundary results obtained in section 3.

Some historical account of the semilinear Calderón problem is in order. The linear Calderón problem on domains in $\mathbb{R}^n$ has been studied intensively. See [16] for a recent survey. The situation has been extended to the case of conformally transversally anisotropic (CTA) manifolds by the authors of [6, 8, 15], but [1, 2] shows that there are Riemannian manifolds for which these methods fail to apply. The successful implementation of complex methods were first employed in solving the Calderón problem in two dimensions [21]. For Riemann surface, based on the work [4], the partial data Calderón problem was solved completely by the authors of [10].

For the semilinear Schrödinger equation $\Delta_g u + V(p, u)$, the two dimensional problem of recovering the potential $V$ was studied in [14, 12], and also in higher dimensional settings [13]. Our method of linearisation is based on the work done fundamentally in [20], and later on generalised in [18, 19] and to CTA manifolds in [5]. In the very recent work [17], the problem has been extended to more general gradient non-linearities. On the other hand, the linearised Calderón problem has been completed solved in the case of real bounded domain as well as on CTA manifolds [7, 9]. In the complex case, the authors of [10, 7] completely solved the partial data linearised Calderón problem on Riemann surface and the full data problem on Stein manifolds.

Throughout this article we will let $d\omega_g$ denote the Riemannian volume element of $(M, g)$ and $d\sigma_g$ the associated boundary element. $\mathcal{C}^s(M)$ will denote the space of Hölder continuous function of order $s$ with the usual topology. For a complex manifold without boundary, we will let $\mathcal{O}(M)$ denote the space of holomorphic functions on $M$.

## 2. Integral Identities

In this section, we reformulate Theorem 1 in terms of a collection of integral identities. The procedure will be similar to that in [18] for the case of real bounded domain, but we prove the result in the form which will be convenient for our purpose.

**Proposition 3.** *Let $(M, g)$ be a compact, oriented manifold, and $\Gamma \subseteq \partial M$ be an arbitrarily small open subset. Let $V, W$ be smooth functions satisfying condition (1) such that the corresponding Dirchlet-to-Neumann partial data maps $\Lambda_V^\Gamma = \Lambda_W^\Gamma$ agree. Assume $V_1 = W_1$ and that for every $f \in \mathcal{C}^\infty(M)$ we* *have*

$$\int_M f u_1 \cdots u_k u_{k+1}\, d\omega_g = 0 \tag{3}$$

*for all $\mathcal{C}^s$ solutions $u_1, ..., u_k$ to the linear Schöringer equation with potential $V_1 = W_1$ and harmonic function $u_{k+1}$, all with boundary data supported in $\Gamma$ implies $f = 0$, then $V = W$.*

It is convenient to formulate the following lemma.

**Lemma 4.** *Assume the setting of Proposition 3. Let $f_\epsilon = \epsilon_1 f_1 + ... + \epsilon_k f_k$ and $\epsilon_1, ..., \epsilon_k$ be small complex parameters, $f_1, ..., f_k$ are in $\mathcal{C}^s(\Gamma)$, $v_{f_\epsilon}$ and $w_{f_\epsilon}$ be solutions to the boundary value problems $(NCP_{f_\epsilon,V})$ and $(NCP_{f_\epsilon,W})$ respectively. If $V_j = W_j$ for all $1 \le j \le k-1$, then for all such $j$ we have*

$$\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_j}}{}_{|_{\epsilon_{\ell_1}=\ldots=\epsilon_{\ell_j}=0}} v_{f_\epsilon} = \partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_j}}{}_{|_{\epsilon_{\ell_1}=\ldots=\epsilon_{\ell_j}=0}} w_{f_\epsilon}$$

*where $\epsilon_{\ell_1}, ..., \epsilon_{\ell_j}$ belong to $\{\epsilon_1, ..., \epsilon_k\}$.*

*Proof.* Since for sufficiently small parameters $\epsilon_1, ..., \epsilon_k$ a unique solution $v_{f_\epsilon}$ to $(NCP_{f_\epsilon,V})$ exists in the class $V_r$ whenever $f_\epsilon \in U_\delta$, we can specify the conditions above. Taking the first order linearisation at zero of the equation

$$\begin{cases} \Delta_g v_{f_\epsilon} + V(p, v_{f_\epsilon}) = 0 & \text{in } M, \\ \qquad\qquad v_{f_\epsilon} = f_\epsilon & \text{on } \partial M \end{cases} \tag{4}$$

with respect to the parameters $\epsilon_1, ..., \epsilon_k$ and using condition (1), we get

$$\begin{cases} \Delta_g \partial_{\epsilon_j}{}_{|_{\epsilon_j=0}} v_{f_\epsilon} = -V_1 \partial_{\epsilon_j}{}_{|_{\epsilon_j=0}} v_{f_\epsilon} & \text{in } M, \\ \quad \partial_{\epsilon_j}{}_{|_{\epsilon_j=0}} v_{f_\epsilon} = f_j & \text{on } \partial M \end{cases}$$

for all $1 \le j \le k$. Thus, $\partial_{\epsilon_j}{}_{|_{\epsilon_j=0}} v_{f_\epsilon}$ solves the linear Schördinger equation with potential $V_1$ and Dirichlet data $f_j$. The same calculation works for the solution $w_{f_\epsilon}$ of $(NCP_{f_\epsilon,W})$. In particular, since $V_1 = W_1$ by assumption, this proves the lemma for $j = 1$ via elliptic regularity. Assume now the claim holds for $j \le k-2$, then we write

$$\Delta_g(v_{f_\epsilon} - w_{f_\epsilon}) = \sum_{j \le k-1} \frac{W_j}{j!}(w^j_{f_\epsilon} - v^j_{f_\epsilon}) + \frac{W_k}{k!} w^k_{f_\epsilon} - \frac{V_k}{k!} v^k_{f_\epsilon} + \sum_{j>k} \frac{W_j}{j!} w^j_{f_\epsilon} - \frac{V_j}{j!} v^j_{f_\epsilon}.$$

Since $v_{f_\epsilon}{}_{|_{\epsilon_1=\ldots=\epsilon_k=0}} = 0$ by estimate (2), taking a $(k-1)$th order linearisation and by considering the terms in $\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}}{}_{|_{\epsilon_{\ell_1}=\ldots=\epsilon_{\ell_{k-1}}=0}} v^j_{f_\epsilon}$ for $j \ge k-1$ which do not contain positive powers of $v_{f_\epsilon}$, we get

$$\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}}{}_{|_{\epsilon_{\ell_1}=\ldots=\epsilon_{\ell_{k-1}}=0}} v^{k-1}_{f_\epsilon} = (k-1)!(\partial_{\epsilon_{\ell_1}}{}_{|_{\epsilon_{\ell_1}=0}} v_{f_\epsilon}) \cdots (\partial_{\epsilon_{\ell_{k-1}}}{}_{|_{\epsilon_{\ell_{k-1}}=0}} v_{f_\epsilon}), \quad \text{and}$$

$$\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}}{}_{|_{\epsilon_{\ell_1}=\ldots=\epsilon_{\ell_{k-1}}=0}} v^j_{f_\epsilon} = 0, \quad j > k+1.$$

On the other hand, for $j \le k-2$ the expression $\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}}{}_{|_{\epsilon_{\ell_1}=\ldots=\epsilon_{\ell_{k-1}}=0}} v^j_{f_\epsilon}$ contains only lower order derivatives of $v_{f_\epsilon}$. The same calculation works for $w_{f_\epsilon}$. Taking these derivatives to $\Delta_g(v_{f_\epsilon} - w_{f_\epsilon})$ and applying elliptic regularity and the inductive hypothesis concludes the proof of the lemma. ☐

Now we prove

**Lemma 5.** *Assume the setting of Proposition 3 and Lemma 4, then for all $k \ge 2$ we have*

$$\sum_{j \le k-1} \int_M \frac{W_j}{j!} u_{k+1} \partial_{\epsilon_1} \cdots \partial_{\epsilon_k}{}_{|_{\epsilon_1=\ldots=\epsilon_k=0}} (v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g + \int_M (V_k - W_k) u_1 u_2 \cdots u_{k+1}\, d\omega_g = 0$$

*where $u_1, ..., u_k$ are $\mathcal{C}^s$ solutions to the linear Schöringer operator with potential $V_1 = W_1$ and $u_{k+1}$ is $\mathcal{C}^s$ harmonic. The Dirichlet data of these solutions are supported in $\Gamma$.*

*Proof.* Let $u_{k+1}$ be a harmonic functions with Dirichlet data $f_{k+1}$ supported in $\Gamma$, then

$$\int_{\partial M} f_{k+1}\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(\mathcal{N}_V^\Gamma - \mathcal{N}_W^\Gamma)f_\epsilon \, d\sigma_g$$

$$= \int_{\partial M\setminus\Gamma} f_{k+1}\partial_\nu\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\sigma_g = 0$$

since $f_{k+1}$ is supported away from the set integrated. The last integral is also equal to

$$-\int_M u_{k+1}\Delta_g\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g + \int_M \langle du_{k+1}, d\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k}(v_{f_\epsilon} - w_{f_\epsilon})\rangle_g \, d\omega_g$$

where

$$\int_M \langle du_{k+1}, d\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k}(v_{f_\epsilon} - w_{f_\epsilon})\rangle_g \, d\omega_g$$

$$= \int_M \partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\Delta_g u_{k+1}\, d\omega_g + \int_{\partial M}\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\partial_\nu u_{k+1} d\sigma_g$$

vanishes. Thus, applying the calculation of Lemma 4, we have

$$\int_M u_{k+1}\Delta_g\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g$$

$$= \int_M u_{k+1}(W_k - V_k)u_1 \ldots u_k \, d\omega_g + \sum_{j\le k-1}\int_M \frac{W_j}{j!}u_{k+1}\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\ldots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g = 0$$

as desired. □

*Proof of Proposition 3.* In virtue of Lemma 4 we will prove the claim via induction on $k$. Notice that for the case $k = 2$, the assumption $V_1 = W_1$ already allows us to invoke Lemma 5 to conclude that

$$\int_M (V_2 - W_2)u_1u_2u_3 \, d\omega_g = 0,$$

so our assumption ensures that $V_2 = W_2$. Now assume the $V_j = W_j$ holds up to some large $j \le k-1$, then combining the statement of Lemma 4 and Lemma 5 again yields that

$$\int_M (V_k - W_k)u_1u_2\cdots u_{k+1}\, d\omega_g = 0.$$

Applying once more our assumption shows that $V_k = W_k$. This concludes the proof of the claim. □

Therefore, in order to solve the semilinear Calderón problem it suffices to prove the assumption in Proposition 3. In the next section we first recover some useful information on the boundary, which will be our key step towards interior identification.

## 3. Boundary Determination

To recover uniqueness in the interior, the first step is often to do so on the boundary. In this section, we will show that the integral identity assumed in Proposition 3 is valid up to infinite order on the boundary. If $(\Sigma, g)$ is a Riemann Surface, then using conformal coordinates the zeroth order result will follow from the proof in the appendix of [10], which will turn out to be enough for our purpose. Thus for the general case it suffices to consider a complex manifold $(M, g)$ with real dimension $\dim M > 2$. For our problem, this is equivalent to assuming that $V_1 = W_1 = 0$, thus the following proposition is sufficient:

**Proposition 6.** *Let $(M, g)$ be a compact, connected Riemannian manifold with smooth boundary. Suppose that $f \in \mathcal{C}^\infty(M)$ satisfy*

$$\int_M fuv \, d\omega_g = 0 \tag{5}$$

*for all $\mathcal{C}^s$ harmonic functions $u, v$ with Dirichlet data supported in in an arbitrarily small open subset $\Gamma \subseteq \partial M$, then $\partial_\nu^k f_{|\Gamma} = 0$ for all $k \in \mathbb{N}$.*

We will follow the idea developed in [3] by exploiting harmonic functions with prescribed boundary data which concentrates at an arbitrary boundary point $p \in \Gamma$ to very fine orders. For this purpose, let $(\chi, \rho) \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1}) \times \mathcal{C}_c^\infty(\mathbb{R})$ be such that

$$\operatorname{Supp}\chi \times \operatorname{Supp}\rho \subseteq B, \quad \|\chi\|_{L^2(\mathbb{R}^{n-1})} = \|\rho\|_{L^2(\mathbb{R})} = 1, \quad \text{where } B \stackrel{\text{def}}{=} (\,|x'| < 1\,) \times (\,0 \le x_n < 1\,)$$

is a relatively open half ball in $\mathbb{R}^n_+$, such that $\chi = \rho = 1$ near the origin and vanish near $\partial B$. For $h > 0$, we also define the notations

$$\chi_h(x') \stackrel{\text{def}}{=} \chi(x'/\sqrt{h}), \quad \rho_h(x_n) \stackrel{\text{def}}{=} \rho(x_n/h), \quad \text{and} \quad B_h \stackrel{\text{def}}{=} (\,|x'| < \sqrt{h}\,) \times (0 \le x_n < \sqrt{h}),$$

where $x' = (x_1, ..., x_{n-1})$ denotes the local boundary coordinates. Without loss of generality, for $h > 0$ small we can assume that $B_h$ lives in a boundary normal coordinate chart of $(M, g)$ centred at $p \in \Gamma$. Thus locally, the metric satisfies

$$g^{\alpha n} = 0, \quad \alpha \le n-1, \quad \text{and} \quad g^{nn} = 1.$$

In particular, we may assume that $g_{ij}(0) = \delta_{ij}$.

We will now proceed to the proof of Proposition 6.

*Proof of Proposition 6.* Following [22], for every $k \in \mathbb{N}$ there exists polynomials $Q_{j,h}$ in $x_n$ such that

$$v_h = \sum_{0 \le j \le 2k+6} h^{j/2} Q_{j,h} e^{(ix'\cdot\tau - x_n)/h}, \quad \text{where}$$
$$Q_{j,h} = \sum_{0 \le i \le j} h^{-i} q_{ij}(x'/\sqrt{h}) x_n^i \quad \text{for all} \quad j \ge 1, \quad Q_{0,h} = \chi_h,$$

where $\tau \in \mathbb{R}^{n-1}$ is a vector locally tangential to $\partial M$, $q_{ij}$ are compactly supported in $(|x'| \le 1)$, and

$$|\Delta_g v_h| = o(h^{k+1}) \quad \text{uniformly on } B_h \text{ as } h \to 0.$$

Consider harmonic functions $u_h = \rho_h v_h + R_h$ where $\|R_h\|_{L^2} \le C\|\Delta_g(\rho_h v_h)\|_{L^2}$. Inputting these functions into the integral identity (3) yields

$$\int_M f|u_h|^2\, d\omega_g = \int_M f|\rho_h v_h|^2\, d\omega_g + \int_M f\rho_h v_h \overline{R}_h\, d\omega_g + \int_M f\rho_h \overline{v}_h R_h\, d\omega_g + \int_M f|R_h|^2\, d\omega_g.$$

Without loss of generality, we may assume that

$$\partial_\nu f_{|\Gamma} = ... = \partial_\nu^{k-1} f_{|\Gamma} = 0,$$

therefore by Taylor's theorem, a careful calculation on the order of $h$ would yield

$$\begin{aligned}
\int_M f|\rho_h v_h|^2\, d\omega_g &= (\partial_\nu^k f)(0) \int_{B_h} x_n^k |\rho_h v_h|^2 \sqrt{\det g}\, dx + o(h^{(n+2k+1)/2}) \\
&= \frac{h^{(n+2k)/2}(\partial_\nu^k f)(0)}{(-2)^k} \int_{|x'| \le 1} |\chi(x')|^2 \int_0^1 |\rho(x_n)|^2 e^{-2x_n/h} \sqrt{\det g}\, dx_n dx' + o(h^{(n+2k+1)/2}) \\
&= \frac{-h^{(n+2k+1)/2}(\partial_\nu^k f)(0)}{(-2)^k} \int_{|x'| \le 1} |\chi(x')|^2 \sqrt{\det g_h(x', 0)}\, dx' + o(h^{(n+2k+1)/2})
\end{aligned}$$

as $h \to 0$. This follows from expanding $v_h$ in the integral, applying convexity and see that

$$h^{j-2i} \int_{|x'| \le \sqrt{h}} |q_{ij}(x'/\sqrt{h})|^2 \int_0^{\sqrt{h}} x_n^{k+2i} |\rho_h|^2 e^{-2x_d/h} \sqrt{\det g}\, dx_n dx' = o(h^{(n+2k+1)/2}), \quad j \ge 1, \ i \le j.$$

Indeed, it suffices to apply integration by parts with respect to the normal direction and change of variables, as well as choosing that $\rho = 0$ in a neighbourhood of $x_n = 1$ to observe this asymptote. Next we look at the $L^2$ estimate of $\Delta(\rho_h v_h)$. By the Leibniz rule, we have

$$\|\Delta_g(\rho_h v_h)\|_{L^2} \le \|\rho_h \Delta_g v_h\|_{L^2} + \|[\rho_h, \Delta_g] v_h\|_{L^2}, \quad \text{where}$$
$$[\rho_h, \Delta_g] v_h = v_h \Delta_g \rho_h + \langle d\rho_h, dv_h \rangle_g$$

is the commutator. By construction it is obvious that $\|\rho_h \Delta_g v_h\|_{L^2} = o(h^{(n+4k+4)/2})$. On the other hand, directly we have

$$\int_M |v_h \Delta_g \rho_h|^2 \, d\omega_g \leq \sum_{j \leq 2k+6} \sum_{i \leq j} C h^{j-2i} \int_{|x'| \leq \sqrt{h}} |q_{ij}(x'/\sqrt{h})|^2 \int_0^{\sqrt{h}} x^{2i} e^{-2x_n} |\Delta_g \rho_h|^2 \, dx_n dx',$$

$$\int_M |\langle d\rho_h, dv_h \rangle|^2 \, d\omega_g \leq \sum_{j \leq 2k+6} \sum_{i \leq j} C h^{j-2i} \int_{|x'| \leq \sqrt{h}} |q_{ij}(x'/\sqrt{h})|^2 \int_0^{\sqrt{h}} |\partial_n x_n^i \partial_n \rho_h|^2 e^{-2x_n/h} \, dx_n dx'$$
$$+ \sum_{j \leq 2k+6} \sum_{i \leq j} C h^{j-2i-2} \int_{|x'| \leq \sqrt{h}} |q_{ij}(x'/\sqrt{h})|^2 \int_0^{\sqrt{h}} |x_n^i \partial_n \rho_h|^2 e^{-2x_n/h} \, dx_n dx'$$

Since the derivatives of $\rho$ vanish at the end points of $[0, \sqrt{h}]$, in all cases the interior integrals in $x_n$ decays to order $o(h^\infty)$ as $h \to 0$ and therefore so do $\|v_h \Delta_g \rho_h\|_{L^2}$ and $\|\langle d\rho_h, dv_h \rangle\|_{L^2}$. It follows from Cauchy-Schwartz that

$$\Big| \int_M f \rho_h v_h \overline{R}_h \, d\omega_g + \int_M f \rho_h \overline{v}_h R_h \, d\omega_g \Big| \leq 2\|f\|_{L^\infty} \|v_h\|_{L^2} \|\Delta_g(\rho_h v_h)\|_{L^2}$$
$$\leq 2\|f\|_{L^\infty} \|v_h\|_{L^2} \|\rho_h \Delta_g v_h\|_{L^2} + 2\|f\|_{L^\infty} \|v_h\|_{L^2} \|[\rho, \Delta_g] v_h\|_{L^2} = o(h^{(n+2k+1)/2}), \quad \text{and}$$
$$\Big| \int_M f |R_h|^2 \, d\omega_g \Big| \leq \|f\|_{L^\infty} \|R_h\|_{L^2}^2 \leq \|f\|_{L^\infty} \|\Delta_g(\rho_h v_h)\|_{L^2}^2 = o(h^{(n+2k+1)/2}).$$

Putting everything together we arrive at

$$\int_M f |u_h|^2 \, d\omega_g = \frac{-h^{(n+2k+1)/2}}{(-2)^k} (\partial_\nu^k f)(0) \int_{|x'| \leq 1} |\chi(x')|^2 \sqrt{\det g_h(x', 0)} \, dx' + o(h^{(n+2k+1)/2}),$$

or in other words

$$0 = (\partial_\nu^k f)(0) \int_{|x'| \leq 1} |\chi(x')|^2 \sqrt{\det g_h(x', 0)} \, dx' + o(1).$$

Taking the limit as $h \to 0$ we thus conclude that $(\partial_\nu^k f)(0) = 0$. Since this holds for every $p \in \Gamma$, the claim follows. □

## 4. Interior Determination

We recall key results on the existence of special holomorphic functions with prescribed critical points and real boundary conditions. The proof of these constructions can be found in [10, 11].

**Proposition 7.** *Let $M$ be a compact complex manifold with smooth boundary. Assume that $M$ has local charts given by global holomorphic functions. Let $k \geq 2$, then we can find a dense subset $S$ of $M$ such that for any $p \in S$, there exists $\Phi \in \mathcal{C}^k(M) \cap \mathcal{O}(Int M)$ having a critical point at $p$ such that both the real part $\varphi$ and imaginary part $\psi$ of $\Phi$ are Morse functions in $M$.*

*In the case of a Riemann surface $\Sigma$, the same is true and we have $\Phi$ is real on $\Gamma^c$, where $\Gamma^c$ is the complement of an arbitrarily small open subset $\Gamma \subseteq \partial\Sigma$. Moreover, for every set of discrete points $\{p, p_1, ..., p_N\}$ we can find holomorphic function $a$ on $\Sigma$ such that $a(p) = 0$ and $a(p_1) = ... = a(p_N) = 0$ up to large orders with the boundary condition that $a_{|\Gamma^c}$ is purely imaginary.*

### 4.1. The Case $n > 1$. Full Data.

Using Proposition 3 and the boundary uniqueness result Proposition 6, the proof of Theorem 1 is now a straightforward application of the result in [11].

*Proof of Theorem 1.* By Proposition 3, it suffices to show that if $f \in \mathcal{C}^\infty(M)$ satisfies condition (3) for all $\mathcal{C}^s$ harmonic functions $u, v$, then $f = 0$. Since we only consider the full data case, by taking identity functions it suffices to replace (3) by (5). Using Proposition 6, we know that in this case $f$ vanishes on $\partial M$ up to infinite orders, therefore we follow the idea in [11] and apply the argument of stationary phase.

Assume that (5) holds. By the results of Proposition 7, there exists a dense subset $S \subseteq M$ such that for every $p \in S$, there exists holomorphic function $\Phi \in \mathcal{C}^k(M) \cap \mathcal{O}(\operatorname{Int} M)$ for some large $k$ such that $p$ is a critical point of $\Phi$ and $\operatorname{Im}\Phi$ is Morse. Choosing

$$u_h = e^{\Phi/h} a \quad \text{and} \quad v_h = e^{-\bar{\Phi}/h}\bar{a}, \quad \forall\, h > 0 \tag{6}$$

where $a \in \mathcal{O}(\operatorname{Int} M)$ satisfies $a(p) = 1$ and $a(p_1) = ... = a(p_N) = 0$ and $\{p, p_2, ..., p_N\}$ is the set of critical points of $\operatorname{Im}\Phi$ in $\operatorname{Int} M$. Such an amplitude obviously can be constructed by the assumption that $M$ is holomorphic separable. In particular, $u_h$ and $v_h$ are respectively holomorphic and anti-holomorphic and so harmonic for all $h > 0$. It follows that we have

$$\int_M f e^{2i\operatorname{Im}\Phi/h} |a|^2 \, d\omega_g = 0, \quad \forall\, h > 0.$$

Take a partition of unity $(\chi_j)$ of $M$ such that $p$ is contained in $\operatorname{Supp}\chi$ but not $\operatorname{Supp}\chi_j$ for any other $j \neq 0$, and $\chi_1(p) = 1$. Since by Proposition 6, $f$ vanishes up to infinite order on $\partial M$, the method of stationary phase yields

$$\big|(2\pi)^n f(p)\sqrt{\det g}(p) \exp\big(\frac{i\pi}{4} \operatorname{Sgn} \nabla_g^2 \operatorname{Im}\Phi(p)\big)(\det\, \nabla_g^2 \operatorname{Im}\Phi(p))^{-1/2}\big| = o(1)$$

as $h \to 0$. Taking this limit we see that $f = 0$ on $S$. Because $S$ is dense in $M$ and $f$ is continuous, $f = 0$ on $M$ as well. □

4.2. **The Case $n = 1$. Partial Data.** We now move on to the consideration of Riemann surface. Using the special structure of having only one complex dimension, adapting the techniques in [10] allows us to prove the partial data result Theorem 2.

*Proof of Theorem 2.* Notice that the Dirchlet-to-Neumann map of $(NCP_{f,V})$ determine the Dirchlet-to-Neumann map of the linear Calderón problem for $V_1$. Indeed, for $\epsilon > 0$ small enough and $\tilde{f} \in \mathcal{C}^s(\Gamma)$ we have that

$$\partial_{\epsilon|_{\epsilon=0}} u_{\epsilon\tilde{f}} + V_1 \partial_{\epsilon|_{\epsilon=0}} u_{\epsilon\tilde{f}} = 0 \ \text{ in } M, \quad \text{and} \quad \partial_{\epsilon|_{\epsilon=0}} u_{\epsilon\tilde{f}} = \tilde{f} \ \text{ on } \Gamma.$$

Thus, if the Dirchlet-to-Neumann maps of $(NCP_{f,V})$ and $(NCP_{f,W})$ are the same, then by the result in [10] we must have $V_1 = W_1 = U$. This reduces the claim to the assumption in Proposition 3. Now, appealing to Proposition 7 once more, we can find a dense subset $S \subseteq \Sigma$ such that for every $p \in S$, we can choose holomorphic function $\Phi \in \mathcal{C}^k(\Sigma) \cap \mathcal{O}(\operatorname{Int}\Sigma)$ for some large $k$ such that $p$ is a critical point of $\Phi$ and $\varphi$, $\psi$ are Morse up to the boundary. Moreover, $\Phi_{|\Gamma^c}$ is purely real, and we can choose $a \in \mathcal{O}(\operatorname{Int}\Sigma)$ such that $a_{|\Gamma^c}$ is purely imaginary, $a(p) = 1$ and $a(p') = 0$ up to arbitrarily large orders for any other critical points $p'$ of $\operatorname{Im}\Phi$. Therefore, choosing

$$\tilde{u}_h = e^{\Phi/h} a + \overline{e^{\Phi/h} a} \quad \text{and} \quad \tilde{v}_h = e^{-\Phi/h} a + \overline{e^{-\Phi/h} a}, \quad \forall h > 0$$

ensures that $\tilde{u}_h$ and $\tilde{v}_h$ are harmonic and

$$\tilde{u}_h = e^{\varphi/h}(\operatorname{Im} a - \operatorname{Im} a) = 0 \quad \text{and} \quad \tilde{v}_h = e^{-\varphi/h}(\operatorname{Im} a - \operatorname{Im} a) = 0 \quad \text{on} \quad \Gamma^c,$$

so that $\operatorname{Supp}\tilde{u}_h$ and $\operatorname{Supp}\tilde{v}_h \subseteq \Gamma$ as well. This is however not quiet enough because we need to extend these solutions to become solutions to the linear Schördinger equation with potential $U$. For that we will assume the technical results proved in Chapter 5 of [10]. We can extend the above via Carleman estimate to become $H^2$ solutions to the Schördinger equation with potential $U$, which have the forms

$$u_h = e^{\Phi/h}(a + ha_0 + r_1) + \overline{e^{\Phi/h}(a + ha_0 + r_1)} + e^{\varphi/h} r_2, \quad \text{and}$$

$$v_h = e^{-\Phi/h}(a + hb_0 + s_1) + \overline{e^{-\Phi/h}(a + hb_0 + s_1)} + e^{-\varphi/h} s_2$$

where $a_0, b_0$ are holomorphic independent of $h$ and the remainder satisfy the properties

$$e^{-\Phi/h}(\Delta_g + V_1) e^{\Phi/h}(a + ha_0 + r_1) = O_{L^2}(h|\log h|), \quad \|r_1\|_{L^2} = O(h),$$

$$\big(e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)}\big)_{|\Gamma^c} = 0, \quad \|r_2\|_{L^2} \leq C h^{3/2} |\log h|.$$

and likewise for $b_0, s_1$ and $s_2$. The Dirichlet boundary data of $u_h$ and $v_h$ can be constructed to have supports on $\Gamma$.

Notice that, the regularities of $u_h$ and $v_h$ as stated are insufficient to be implemented into (3). For this we consider a sequence of smooth approximations $(f_{h,j})_j$ defined on the boundary such that $\operatorname{Supp} f_{h,j} \subseteq \Gamma$ and $\lim_{j\to\infty} f_{h,j} = u_{h|\partial M}$ in $H^1(\partial\Sigma)$. Let $(\phi_{h,j})_j$ be the corresponding smooth solutions to $(\Delta_g + U)u = 0$, $u_{|\partial\Sigma} = f_{h,j}$. Then ellptic boundary regularity estimates ensure that $\lim_{j\to\infty} \phi_{h,j} = u_h$ in $H^1$. By making a similar calculation for $v_h$, we conclude that there exists smooth approximations $(\psi_{h,j})_j$ and remainders $R_{u_h}$, $R_{v_h}$ depending on $j$ such that

$$\phi_{h,j} = u_h + R_{u_h}, \quad \psi_{h,j} = v_h + R_{v_h}, \quad \text{where} \quad \lim_{j\to\infty} R_{u_h} = \lim_{j\to\infty} R_{v_h} = 0$$

in $H^1$. Assume first we have

$$\int_\Sigma fuv\, d\omega_g = 0$$

for all $\mathcal{C}^s$ solutions to the Schördinger equation with Dirichlet data supported on $\Gamma$ as in (3). Then

$$0 = \int_\Sigma f\phi_{h,j}\varphi_{h,j}\, d\omega_g = \int_\Sigma fu_h v_h\, d\omega_g + \int_\Sigma fu_h R_{v_h}\, d\omega_g + \int_\Sigma fv_h R_{u_h}\, d\omega_g + \int_\Sigma R_{u_h} R_{v_h}\, d\omega_g.$$

The later integrals converge to zero as $j \to \infty$. By taking this limit we thus arrive at

$$\int_\Sigma fu_h v_h\, d\omega_g = 0.$$

Now we get an expansion

$$0 = I_1 + I_2 + o(h),$$

for

$$I_1 \stackrel{\text{def}}{=} \int_\Sigma f(a^2 + \overline{a}^2)\, d\omega_g + 2\mathrm{Re} \int_\Sigma e^{2i\psi/h} f|a|^2\, d\omega_g, \quad \text{and}$$

$$I_2 \stackrel{\text{def}}{=} 2h\mathrm{Re} \int_\Sigma af\big(e^{2i\psi/h}\overline{(\frac{s_1}{h} + b_0)} + e^{-2i\psi/h}\overline{(a_0 + \frac{r_1}{h})} + b_0 + a_0 + \frac{s_1 + r_1}{h}\big)\, d\omega_g$$

Applying the theorem of stationary phase by means in [10], we have

$$0 = 2Cf(p)|a(p)|^2 + o(1), \quad C \neq 0$$

as $h \to 0$. Hence $f = 0$.

To prove the general case, we assume that

$$\int_M fuvw\, d\omega_g = 0$$

for all $\mathcal{C}^s$ solutions $u, v$ to the Schördinger equation and harmonic functions $w$ with Dirichlet data supported on $\Gamma$. If $w$ is smooth, then by what was proved we deduce $fw = 0$, thus

$$\int_\Sigma fuv\, d\omega_g = 0$$

for all smooth harmonic functions $u$ and $v$ with Dirichlet data supported in $\Gamma$. In virtue of the smooth approximation argument above, this implies $f = 0$. Now suppose inductively

$$\int_\Sigma fu_1 \cdots u_{k-1} w\, d\omega_g = 0$$

for all $\mathcal{C}^s$ solutions $u_1, ..., u_{k-1}$ to the Schórdinger equation and harmonic functions $w$ implies $f = 0$, and assume

$$\int_\Sigma fu_1 \cdots u_{k-1} u_k w\, d\omega_g = 0$$

for an addition $\mathcal{C}^\infty$ solution to the Schórdinger equation as above. Choosing $u_k$ smooth again implies $fu_{k-1} = 0$ and another smooth approximation argument yields $f = 0$. This concludes the proof of the claim. □

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(Y. Ma) F07-Carslaw BUILDING, THE UNIVERSITY OF SYDNEY, CHIPPENDALE, NSW, AUSTRALIA
*Email address*: K.Ma@maths.usyd.edu.au

(L. Tzou) F07-Carslaw BUILDING, THE UNIVERSITY OF SYDNEY, CHIPPENDALE, NSW, AUSTRALIA
*Email address*: leo.tzou@sydney.edu.au