Dedicated to Yuchen, who remained by my side in spite of my unpredictable temper and continuous absences.

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Preface

This document is written in fulfilment of my honour degree in pure mathematics at the University of Sydney. Our initial intention was to give a complete treatment of the *Kardar–Parisi–Zhang equation*, which interested me because of its popularity after the year 2014, during which a fields medallist level work was associated with it. The equation also had significant relations with the notion of universality classes in probability, which I always found to be quiet fascinating. In order to restrict ourselves to topics in analysis, we decided to focus on an alternative of the original theory, nowadays known as the method of paracontrolled distributions, which relies on nothing more than well-known stochastic and Fourier analytic techniques. However, gradually we realised that the size of this project was too ambitious for a thesis which was limited to only 60 pages, and so in the final product we were forced to remove a number of interesting probabilistic topics, but only presented the functional analytic pillars of the arguments. As a consequence, the reading of this thesis requires almost no prerequisites in stochastic analysis. What remains here is the result of that process.

We made efforts to introduce the theory with as much motivations as possible. In the introductory chapter we briefly look at some big pictures on singular stochastic partial differential equations, beginning from the very first theorem in the field. We paid particular attentions to how and why various ideas emerged by trying to understand the motivations of authors who wrote evolutionary papers during different times. In Chapter One we introduce the Fourier analytic tools which will be made use of. These will be the most technical aspects of this thesis. Starting from Chapter Two we will look at ordinary differential equations where the situation is very clear. In Chapter Three we will move on to partial differential equations, where the theory there is historically built on those of the ordinary cases.

I would like to thank first and foremost Professor Beniamin Goldys, who patiently supervised me in writing this thesis, while being the mentor of my academic life as well. From time to time I found myself reliant on his presences in order to move on from difficulties, some of them I suspect were beyond his responsibilities, yet he guided me through them nonetheless, and for that I am truly grateful. I would also like to thank those who lectured me through my undergraduate and honour curriculum, including but not limited to F.C. Cirstea, A. Fish, L. Tzou, Z. Zhang, H. Wu, U. Keich, M. Wechselberger and R. Marangell. I would like to thank in particular Professor Daniel Daners, who single handedly constructed the curriculum in mathematical analysis for the School of Mathematics and Statistics at the University of Sydney. The elegance in his proofs have been the main factor for my choice to study mathematical analysis in depths; and Professor Laurentiu Paunescu, who acted as the honour coordinator and assisted me with everything I ever needed as a honour student. Finally, I would like to thank my family and friends who supported me financially throughout the years. I would not have not survived without them. I would also like to thank all those baristas who sold me plenty of coffees as well.

Chapter 1

Introduction

The area of stochastic partial differential equations (*SPDEs*) is an active field of research where problems are often heavily related to the physical sciences. Since experiments are often probabilistic in nature, mathematical models which capture randomness are sometimes looked in favour over the deterministic ones. However, whereas mathematicians take considerations of well-posedness, scientists working in the general discipline pose their equations caring primarily the capacity of their models in explaining experimental results. As a consequence, stochastic equations explaining fundamental phenomenons in nature which are very well-established in physics turn out to be complete nonsense in the language of mathematics.

In the late nineteenth century there had been many such equations, and it was apparent that the existing tools in stochastic analysis were insufficient to give them meanings. Such problems arise mostly because the graph of sufficiently well-behaved random signals most also be quiet rough. For an example, consider the one-dimensional *Kardar–Parisi–Zhang (KPZ) equation*:

$$\partial_t h(x,t,\omega) - \nu\partial_x^2 h(x,t,\omega) = \frac{\lambda}{2}(\partial_x h)^2(x,t,\omega) + \sqrt{D}\eta(x,t,\omega),$$

where $\eta : \mathbb{R}\times(0,\infty)\times\Omega \to \mathbb{R}$ is the stochastic *Gaussian space-time white noise* for some suitable probability space (Ω, P), the constants $(\nu, \lambda, D) \in \mathbb{R}^2 \times (0,\infty)$ are parameters and h is continuous in both space and time. Named after its creators M. Kardar, G. Parisi and Y. Cheng, the equation was first applied to model random interface growth and has gained enough popularity to become the default setting for the topic.

Mathematically, for every possible out come $\omega \in \Omega$, the smoothness of the solution $h(\omega)$ is inevitably dependent on the smoothness of $\eta(\omega)$, which is known to be very rough, one could show that $h(\omega)$ is at most Hölder continuous for indices $s < 1/2$. It is then a technical matter to describe what it is meant of the derivative $\partial_x h(\omega)$, which turns out to require the introduction of an entirely new class of elements known as tempered distributions. In particular, the product $(\partial_x h)^2(\omega)$ cannot even be interpreted as the product of functions, which means the the *KPZ* equation is a priori ill-defined. Then centre of the problem now concerns the establishment of a working framework in which one could discuss the *KPZ* equation meaningfully.

The Theory of Rough Path

It was in the 1990s when British mathematician T.J. Lyons first considered a deterministic approach to solving stochastic differential equations

$$du(t,\omega) = f(u)(t,\omega)dB(t,\omega), \quad u(0,\omega) = u_0(\omega)$$

for the Brownian motion B and a $u : \mathbb{R} \times \Omega$ is continuous in time. In the probabilistic literature, such an equation is usually interpreted as the *Stratonovich or Ito's integral equation*:

$$u(t, \omega) = u_0(\omega) + \int_0^t f(u)(s, \omega) dB(s, \omega), \tag{1.1}$$

where the pathwise regularity of $B(\omega)$ satisfies the condition

$$\|B(\omega)\|_{\mathcal{V}^p(I)} \stackrel{\text{def}}{=} \left(\sup_{\mathcal{P} \subset I} \sum_{t_j \in \mathcal{P}} |B(t_j, \omega) - B(t_{j+1}, \omega)|^p \right)^{1/p} < \infty, \quad p > 2,$$

and the surepmum is taken over all partitions $\mathcal{P} = (t_j)_j$ of I. In other words, $B(\omega)$ has finite p variation for all $p > 2$. Such a condition deems that the integral (1.1) cannot be finite for every fixed $\omega \in \Omega$, and Lyons wanted to know whether it was possible to make sense of this integral and to what extend the corresponding differential equation is well-posed, as at his time this integrability condition was already pushed to the boundary case by L.C.Young, who proved that

$$\int_0^t u(s) dX(s) \stackrel{\text{def}}{=} \lim_{|\mathcal{P}| \to 0} \sum_{t_j \in \mathcal{P}} u(t_j)(X(t_{j+1}) - X(t_j)) \;\; \text{exists, provided}$$

$$\|u\|_{\mathcal{V}^q(I)}, \|X\|_{\mathcal{V}^p(I)} < \infty, \quad \text{for all } q, p \text{ such that } \; 1/q + 1/p > 1.$$

Here $|\mathcal{P}|$ is the mesh of $\mathcal{P}$. It turns out this is doable provided we restrict out attention to a suitable space of *geometric rough paths*, for which we denote by $\mathcal{X}$. This construction by Lyons is algebraic in nature and somewhat very technical. More precisely, let Δ_T be the simplex

$$\Delta_T \stackrel{\text{def}}{=} \{(s,t) \in [0,T] \mid 0 \leq s \leq t \leq T\}, \quad T > 0.$$

Lyons observed that for every continuous path $x : [0,T] \to \mathbb{R}^n$ and $P \geq 1$, it is sufficient to study information of x stored in its *step-$\lfloor p \rfloor$ signature*:

$$X_{\lfloor p \rfloor} : \Delta_T \to T^{\lfloor p \rfloor}(\mathbb{R}^n), \quad T^{\lfloor p \rfloor}(\mathbb{R}^n) \stackrel{\text{def}}{=} \mathbb{R} \oplus \bigoplus_{1 \leq j \leq \lfloor p \rfloor} (\mathbb{R}^n)^{\otimes j},$$

$$X_{\lfloor p \rfloor}(s,t) \stackrel{\text{def}}{=} (1, X^{(1)}(s,t), ..., X^{(\lfloor p \rfloor)}(s,t)) \in \mathbb{R} \otimes \mathbb{R}^n \otimes (\mathbb{R}^n)^{\otimes 2} \otimes ... \otimes (\mathbb{R}^n)^{\otimes \lfloor p \rfloor},$$

where the elements in the tensor products are of the form

$$X^{(k)}(s,t) \stackrel{\text{def}}{=} \sum_{1 \leq i_1, ..., i_k \leq n} \left(\int_{s < u_1 < ... < u_k < t} dx_{u_1}^{i_1} ... dx_{u_k}^{i_k} \right) (e_{i_1} \otimes ... \otimes e_{i_k})$$

A map $X : \Delta_T \to T^{\lfloor p \rfloor}(\mathbb{R}^n)$ is then in $\mathcal{X}$ if there exists a sequence of such continuous maps $(x(m))_{m \geq 1}$ with finite 1-variation such that $\lim_{m \to \infty} X_{\lfloor p \rfloor}(m) = X$ in the *$\lfloor p \rfloor$-th variation metric*:

$$d_p(X, X_{\lfloor p \rfloor}(m)) \stackrel{\text{def}}{=} \max_{1 \leq k \leq \lfloor p \rfloor} \sup_{\mathcal{P} \subset [0,T]} \left(\sum_{t_j \in \mathcal{P}} |X^{(k)}(t_{j-1}, t_j) - X^{(k)}_{\lfloor p \rfloor}(m)(t_{j-1}, t_j)|^{p/k} \right)^{1/p}.$$

Occasionally a geometric rough path is also denoted by $\mathbb{X}$. For every p-geometric rough path Lyons defined what is meant by (1.1). Moreover, the pathwise Brownian motion $B(\omega)$ is a p-geometric rough path for all $2 < p < 3$. To this end, Lyons also gave a meaning to the pathwise solution of the stochastic differential equation (1.1), which he calls the *Universal Limit Theorem*:

Theorem 1. *Let X be a p-geometric path and $(x(m))_{m\in\mathbb{N}}$ be a sequence of continuous paths with finite 1-variation with $\lfloor p \rfloor$-step signatures $X_{\lfloor p \rfloor}(m)$. Let $f : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)$ be a vector field with at least $\lfloor p \rfloor$ bounded derivatives and is Hölder continuous of order $s > p$. Let $u(m)$ be the solution to the ordinary differential equation*

$$du(m)(t) = f((u))(m)(t)dX(m)(t), \quad u(m)_{|t=0} = u_0(m)$$

with $\lim_{m\to\infty} u_0(m) = u_0$. Then the sequence of step-$\lfloor p \rfloor$ signature $u_{\lfloor p \rfloor}(m)$ converges to to a p-geometric rough path u in the p-variation metric. Moreover, u solves the differential equation

$$du(t) = f(u)(t)dX(t), \quad u_{|t=0} = u_0$$

driven by the p-geometric rough path X.

We will prove the theory of Lyons under different frameworks in Chapter Two. For a detailed treatment of the theory of rough path, see [5].

A Fourier Analytic Formulation

The theory of rough path gave a first example of special structures to solutions of equations with stochastic noise. The weakness in the theory being that its formulation in terms of path is specialised to work only for ordinary differential equation. In the meanwhile, it hinted the fact that when dealing with equations with rough signals, one cannot always expect to make sense of the equation, and additional constraints must postulated.

Following Lyon's footsteps, Italian mathematician Massimiliano Gubinelli and his collaborators were successful in reformulateing Lyons' idea in the language of Fourier analysis and moreover, generalised it to frameworks which were applicable to *SPDEs*. Taking advantages of the theory of distributions and the fact that the topology induced by the p-variation norm correspond exactly to the $1/p$-Hölder topology. The theorem of Lyons is equivalent to finding the correct structure of the solution u to the *rough differential equation*:

$$\frac{d}{dt}u(t) = f(u(t))dX(t), \quad u_{|t=0} = u_0$$

for rough signals X which is Hölder continuous

$$\|X\|_{C_b^{0,s}} \stackrel{\text{def}}{=} \|u\|_{L^\infty} + \sup_{t\neq t'} \frac{|X(t) - X(t')|}{|t-t'|^s} < \infty, \quad \forall s < 1/2,$$

and satisfying some approximation condition similar to geometric rough paths, such that the non-linearity $f(u)dX$ is a priori well-defined and the solution u depends Lipschitz continuously on the rough signal X and initial condition u_0. One natural Fourier analytic instrument for this is known as *the Paley-Littlewood theory*, which gives a natural meaning of products of distributions in terms of a decomposition

$$f(u(t))dX(t) = f_1(u(t), X(t)) + f_2(u(t), X(t), R(X(t), dX(t))),$$

where $f_1(u(t), X(t))$ is well-defined with the natural regularity condition, and that $f_2(u(t), X(t), R(X(t), dX(t)))$ makes sense if and only if $X(t)$ and $R(X(t), dX(t))$ can be approximated via smooth functions $(X^\epsilon(t), R(X^\epsilon(t), dX^\epsilon(t)))_{\epsilon>0}$ in the product Hölder topology for some function R. In that case, Gubinelli proved that the map $(X, dX, u_0) \to u$ satisfies the required Lipschitz conditions.

This approach is advantageous in the sense that it is no longer restrictive to ordinary differential equations. It turns out for *SPDEs* as complicated as the *KPZ* equation, it is still possible to find the right structure

$$h = h_{\text{finite}} + h_{\text{singular}}, \quad \text{and} \quad S(\eta(\omega)) = (S_1(\eta(\omega)), ..., S_m(\eta(\omega)))$$

such that $(\partial_x h)^2$ becomes well-defined provided h satisfies the given structure, and that the pathwise stochastic signals $S_1(\eta(\omega), ..., S_m(\eta(\omega)))$ admit smooth approximating sequences $(S_1^\epsilon(\eta(\omega)), ..., S_m^\epsilon(\eta(\omega)))_{\epsilon>0}$ which converge to $S(\eta(\omega))$ in a suitable topology as ϵ goes to zero. The idea is that we isolate the problematic term h_{singular}, which causes the non-linearity to be ill-defined but can be fixed under the influence of $S(\eta(\omega))$. The trade-off being that by postulating these conditions, one also needs to prove the existence and uniqueness of such a h, which eventually must solve the *KPZ* equation. The required approximation of $S(\eta(\omega))$ must also be explicitly construable using stochastic analysis as well.

For general singular stochastic equations with some ill-defined non-linearities, one thus needs to consider:

- which elements of the stochastic signal need to be approximated by smooth functions;

- what structure of the solution can isolate the singularity and whether such singularity is fixable by postulating the above stochastic approximations;

- whether these restrictions are appropriate such that a fixed point argument can be closed, or that the stochastic approximations can be concretely constructed via stochastic analysis.

Since these structures are usually found by applying paradifferential calculus, such an approach is also known to be *the theory of paracontrolled distributions*. Chapter One of this text will be concerned with introducing the necessary analytic tools and we will demonstrate this method on the *KPZ* equation in Chapter Three.

Other Generalisations: The Theory of Regularity Structures

Finally, it is worth noting that there exists more than one generalisation to the theory of rough path. Most significantly, Swiss mathematician M. Hairer independently developed what is now known as the *theory of regularity structures*. Prior to the method of paracontrolled distributions, Hairer gave the first formulation of a robust solution theory to the *KPZ* equation via regularity structures and was subsequently awarded the fields medal in 2014. See [9] and the references thereafter for more details. Many correspondences between the method of paracontrolled distributions and the theory of regularity structures have been shown, see [2]. To this date, researches are still consistently being conducted in the area.

Before we begin our formal discussion, we make the remark that since we will be dealing with a great number of estimates, throughout the text we will generally use C to denote any strictly positive constants, with subscripts indicating dependencies.

Chapter 2

Analysis in Besov Spaces

In this chapter we introduce the appropriate function spaces where our analysis of partial differential equations will take place in. Although the theory of Besov space is quiet overreaching, we will only be concerned with the special case which generalises the spaces of Hölder continuous function. We will start with the basic tool that will be used throughout this text, that is the Fourier transform.

2.1 Fourier Transform on Tempered Distributions

In this section we will recall some elementary facts of Fourier transform on the space of *Schwartz functions*. As an extension we will define the space of *tempered distributions* and the corresponding *Fourier transform* that follows. We define $\mathcal{D}(\mathbb{R}^n)$ to be the space of all compactly supported complex valued smooth functions ϕ with continuous derivatives of all orders. If Ω is a subset of $\mathbb{R}^n$, then we also let $\mathcal{D}(\Omega)$ denote the space of all functions in $\mathcal{D}(\mathbb{R}^n)$ which are compactly supported in the closure of Ω. The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth functions $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that

$$\|\phi\|_{k,\mathcal{S}} \stackrel{\text{def}}{=} \sup_{|\alpha|\le k} \|(1+|\cdot|^2)^k \partial^\alpha u\|_{L^\infty} < \infty, \quad k \in \mathbb{N}.$$

It is straightforward to check that, the topology generated by the family of seminorms $(\|\cdot\|_{k,\mathcal{S}})_{k\in\mathbb{N}}$ is equivalent to the topology generated by a Fréchet space. Thus, linear operators defined on $\mathcal{S}(\mathbb{R}^n)$ are continuous if and only if they are bounded. In particular:

1. If $L : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a linear functional, then L is continuous if and only if there exists integer k such that

   $$|\langle L, \phi\rangle| \le C\|\phi\|_{k,\mathcal{S}} \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n),$$

   where we follow the notation that $\langle L, \phi\rangle$ denotes the application of L on ϕ.

2. If $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear operator, then L is continuous if and only if for each integer k, there exists a integer N such that

   $$\|L\phi\|_{k,\mathcal{S}} \le C\|\phi\|_{N,\mathcal{S}}.$$

Recall that the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{F}\phi(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \phi(x) e^{-\xi\cdot x} dx.$$

It is also customary to denote such a operation by $\widehat{\phi}(\xi)$. Moreover, by the classical results in Fourier analysis, the mapping $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear homeomorphism with inverse

$$\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi(x) e^{-\xi \cdot x} d\xi.$$

Using these information, we can describe tempered distributions.

Definition 2.1.1. A tempered distribution on $\mathbb{R}^n$ is any continuous linear functional u on $\mathcal{S}(\mathbb{R}^n)$. The space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

We equip $\mathcal{S}'(\mathbb{R}^n)$ with the weak-$*$ topology with respect to $\mathcal{S}(\mathbb{R}^n)$. Thus, a sequence $(u_n)_{n\in\mathbb{N}}$ of tempered distributions converges to u in $\mathcal{S}'(\mathbb{R}^n)$ if and only if

$$\forall \phi \in \mathcal{S}(\mathbb{R}^n), \quad \lim_{n\to\infty} \langle u_n, \phi\rangle = \langle u, \phi\rangle.$$

It is most convenient to define linear operators on tempered distributions via duality.

Proposition 2.1.2. *Let L be a linear continuous map from $\mathcal{S}$ into $\mathcal{S}$. The formula*

$$\langle {}^tLu, \phi\rangle \stackrel{def}{=} \langle u, L\phi\rangle$$

then defines a tempered distribution. Moreover, ${}^tL$ is sequential continuous.

Proof. By assumption, the linear map L is continuous, hence there exists a constant C and an integer N such that

$$\forall \phi \in \mathcal{S}, \quad \|L\phi\|_{k,\mathcal{S}} \le C\|\phi\|_{N,\mathcal{S}}.$$

Thus if u is a tempered distribution, then an integer k and a constant C' exists such that

$$\forall \phi \in \mathcal{S}, \quad |\langle {}^tLu, \phi\rangle| = |\langle u, L\phi\rangle| \le C'\|L\phi\|_{k,\mathcal{S}} \le C\|\phi\|_{N,\mathcal{S}}.$$

The sequential continuity now follows from

$$\langle {}^tLu_n, \phi\rangle = \langle u_n, L\phi\rangle \longrightarrow \langle u, L\phi\rangle = \langle {}^tLu, \phi\rangle.$$

The proposition is thus proved. □

Let us now provide two important examples of tempered distributions.

Example 2.1.3. The space L^1_{loc} of locally integrable functions f on $\mathbb{R}^n$ such that $(1 + |x|^2)^{-N} f(x) \in L^1$ for some $N \in \mathbb{N}$ can be identified with a subset of $\mathcal{S}'$ by the formula

$$L^1_{loc} \to \mathcal{S}' : f \longmapsto T_f, \quad \text{with} \quad \langle T_f, \phi\rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

Example 2.1.4. Any finite Borel measure $\mu \in \mathcal{M}_{\mathcal{B}}$ can be identified with an element of $\mathcal{S}'$ by the formula

$$\mathcal{M}_{\mathcal{B}} \to \mathcal{S}' : \mu \longmapsto T_\mu, \quad \text{with} \quad \langle T_\mu, \phi\rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \phi(x)d\mu.$$

We now list a few situations where Proposition 2.1.2 applies.

Example 2.1.5. The differential and multiplication operators $-\partial^\alpha$ or $\mathcal{M}_{x^\alpha}$ with $\alpha \in \mathbb{N}^n$ are continuous since

$$\forall \phi \in \mathcal{S}, \quad \|(-\partial)^\alpha \phi\|_{k,\mathcal{S}} \leq \|\phi\|_{k+|\alpha|,\mathcal{S}} \quad \text{and} \quad \|\mathcal{M}_{x^\alpha}\phi\|_{k\mathcal{S}} = \|x^\alpha \phi\|_{k,\mathcal{S}} \leq \|\phi\|_{k+|\alpha|,\mathcal{S}}.$$

Thus the corresponding dual operators ${}^t(-\partial^\alpha)$, ${}^t\mathcal{M}_{x^\alpha}$ are continuous linear maps on tempered distributions. Similarly, if f is a smooth function, then an integer N exists such that

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^k)^{-N} \sup_{|\alpha| \leq k} |\partial^\alpha f(x)| < \infty.$$

Thus, the multiplications ${}^t\mathcal{M}_f : \mathcal{S}' \to \mathcal{S}'$ defines linear operators on tempered distribution as well.

Example 2.1.6. Let θ be a Schwartz function, the *convolution* between θ and $\phi \in \mathcal{S}$ is

$$\theta * \phi \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \theta(x-y)\phi(y)dy.$$

Because of the identity

$$\forall \phi \in \mathcal{S}, \quad \|\check{\theta} * \varphi\|_{k,\mathcal{S}} \leq C_k \|\theta\|_{k+n+1,\mathcal{S}} \|\varphi\|_{k,\mathcal{S}}, \quad \check{\theta} \stackrel{\text{def}}{=} \theta(-\cdot)$$

the operation $\phi \mapsto \check{\theta} * \phi$ is continuous. Thus the formula

$$\forall \phi \in \mathcal{S}, \quad \langle \theta * u, \phi \rangle \stackrel{\text{def}}{=} \langle u, \check{\theta} * \phi \rangle$$

defines a linear operator on tempered distribution. Hence convolution is defined on $\mathcal{S}'$.

Example 2.1.7. In view of the continuity of $\mathcal{F}$ on $\mathcal{S}$, the formula

$$\forall \phi \in \mathcal{S}, \quad \langle {}^t\mathcal{F}u, \phi \rangle \stackrel{\text{def}}{=} \langle u, \mathcal{F}\phi \rangle$$

defines a linear opeartor on tempered distribution.

Example 2.1.8. Let A be a linear automorphism of $\mathbb{R}^n$ and define

$$L_A \phi \stackrel{\text{def}}{=} \frac{1}{\det(A)} \phi \circ A^{-1}.$$

Then L_A also satisfies the hypothesis of Proposition 2.1.2.

In order to extend classical operations to tempered distributions, in what follows we will make the abuse of notation and write ∂^α, $\mathcal{M}_\theta$, L_A and $\mathcal{F}$ as linear mapping $\mathcal{S}' \to \mathcal{S}'$ respectively in places of ${}^t(-\partial^\alpha)$, ${}^t\mathcal{M}_\theta$, ${}^tL_A$ and ${}^t\mathcal{F}$.

The Fourier transform on tempered distributions allows for many classical properties defined on $\mathcal{S}$ to be transferred to the case of $\mathcal{S}'$.

Proposition 2.1.9. *For any $(u, \theta) \in \mathcal{S}' \times \mathcal{S}$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $(a, \omega) \in \mathbb{R}^n \times \mathbb{R}^n$, we have*[1]

$$\partial^\alpha \hat{u} = \mathcal{F}\Big((ix)^\alpha u\Big), \quad (i\xi)^\alpha \hat{u} = \mathcal{F}(\partial^\alpha u), \quad e^{-ia\cdot\xi}\hat{u} = \mathcal{F}(\tau_a u),$$
$$\tau_\omega \hat{u} = \mathcal{F}(e^{ix\cdot\omega}u), \quad \lambda^{-n}\hat{u}(\lambda^{-1}\xi) = \mathcal{F}\Big(u(\lambda\cdot)\Big), \quad \text{and} \quad \mathcal{F}(\theta * u) = \hat{\theta}\hat{u}.$$

[1] Here we let τ_a denotes translation by $-a$ for arbitrary $a \in \mathbb{R}^n$.

Proof. The first five properties follow obviously from their counterpart on $\mathcal{S}$. To show the convolution theorem on $\mathcal{S}'$, notice that we have

$$\langle \mathcal{F}(\theta * u), \phi\rangle = \langle u, \check{\theta} * \mathcal{F}\phi\rangle.$$

By the simple calculation

$$\begin{aligned}\check{\theta} * \mathcal{F}\phi &= \int_{\mathbb{R}^n} \theta(\eta-\xi)\Big(\int_{\mathbb{R}^n} \phi(x)e^{-ix\cdot\eta}dx\Big)d\eta \\ &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\Big(\int_{\mathbb{R}^n} e^{-i(x\cdot\eta-\xi)}\theta(\eta-\xi)d\eta\Big)\phi(x)dx = \mathcal{F}(\hat{\theta}\phi)\end{aligned}$$

we thus infer that

$$\langle \mathcal{F}(\theta * u), \phi\rangle = \langle u, \mathcal{F}\Big(\hat{\theta}\phi\Big)\rangle = \langle \hat{u}, \hat{\theta}\phi\rangle = \langle \hat{\theta}\hat{u}, \phi\rangle$$

which proves the result □

2.2 Dyadic Partition of Unity

We now define a special kind of partition of unity, which will become immensely useful for our theory throughout the rest of this text.

Proposition 2.2.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n \mid 3/4 \le |\xi| \le 8/3\}$. There exists radial functions χ and φ, valued in $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and $\mathcal{D}(\mathcal{C})$, and such that*

$$\forall \xi \in \mathbb{R}^n\ ,\ \chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \tag{2.1}$$

$$|j-j'| \ge 2 \Rightarrow \text{Supp }\varphi(2^{-j}\cdot)\cap \text{Supp }\varphi(2^{-j'}\cdot) = \emptyset, \tag{2.2}$$

$$j \ge 1 \Rightarrow \text{Supp }\chi \cap \text{Supp }\varphi(2^{-j}\cdot) = \emptyset. \tag{2.3}$$

Proof. Take α in $(1, 4/3)$ and denote by $\tilde{\mathcal{C}}$ the annulus with small radius α^{-1} and big radius 2α. Use the existence of radial smooth cut-off function to choose θ valued in $[0,1]$, with value 1 in a neighbourhood of $\tilde{\mathcal{C}}$. Without loss of generality assume $j' \ge j$, then

$$2^{j'}\mathcal{C} \cap 2^j\mathcal{C} \ne \emptyset \Rightarrow 2^{j'} \times 3/4 \le 4 \times 2^{j+1}/3 \quad \text{and} \quad j'-j \le 1.$$

Therefore

$$|j-j'| \ge 2 \Rightarrow 2^{j'}\mathcal{C} \cap 2^j\mathcal{C} = \emptyset. \tag{2.4}$$

Let

$$S(\xi) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \theta(2^j\xi) \quad \text{and} \quad \varphi(\xi) \stackrel{\text{def}}{=} \theta(\xi)/S(\xi)\ ,\ \chi(\xi) \stackrel{\text{def}}{=} 1 - \sum_{j\ge 0}\varphi(2^{-j}\xi).$$

With these definitions, Supp $\varphi(2^j\cdot) \subseteq 2^j\mathcal{C}$ and Supp $\chi \subseteq B(0, 4.3)$. (2.4) then ensures that S is locally finite and so smooth, and (2.2), (2.3) are also satisfied. Finally

$$\chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1 - \sum_{j\ge 0}\varphi(2^{-j}\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1,$$

so that (2.1), and thus our claim is proved. □

The *Fourier multiplier with symbol* $\theta(a\xi)$ is defined for every real number and $\theta \in \mathcal{D}$ the unique function which satisfies

$$\mathcal{F}\theta(aD) = \theta(a\xi)\hat{u}.$$

We now let $u \in \mathcal{S}(\mathbb{R}^n)$ and set

$$\Delta_j u \stackrel{\text{def}}{=} 0 \quad \text{if} \quad j \leq -2, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \mathcal{F}^{-1}\chi * u,$$
$$\text{and} \quad \Delta_j u \stackrel{\text{def}}{=} \varphi(2^{-j}D)u = \mathcal{F}^{-1}\varphi(2^{-j}\cdot) * u \quad \text{if} \quad j \geq 0.$$

The mappings $(\Delta_j)_{j\geq -1}$ are *Littlewood-Paley blocks* or simply dyadic blocks. Moreover, the *nonhomogenous low-frequency cut-off operator* is the mapping

$$S_j u \stackrel{\text{def}}{=} \sum_{j' \leq j-1} \Delta_{j'} u = \chi(2^{-j}D)u, \quad j \geq 0$$

By letting $h_j = \mathcal{F}^{-1}\varphi(2^{-j}\cdot)$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, we also have

$$\Delta_j u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x-y)dy \quad \text{and} \quad \Delta_{-1} u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y)u(x-y)dy.$$

Similarly, we let

$$S_j u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}\Big(2^j y\Big)u(x-y)dy.$$

Remark 2.2.2. The dyadic partition of unity constructed in Proposition (2.2.1) is clearly not unique. In general, any (χ, φ) in $\mathcal{D}(B') \times \mathcal{D}(\mathcal{C}')$ which satisfies (2.1) and almost disjointness in the sense of (2.2) are dyadic partitions of unity. Since for every $u \in L^\infty(\mathbb{R}^n)$ and $j \geq 0$ we have

$$\sup_{x\in\mathbb{R}^n} \left| 2^{jn} \int_{\mathbb{R}^n} h\Big(2^j(x-y)\Big)u(y)dy \right| \leq \|u\|_{L^\infty} \|h\|_{L^1}$$

and likewise for the case of $\tilde{h}$. The operators $\Delta_j u$ and S_j map $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ continuously for all of j.

The point of the above construction is to justify the following convergence

$$\text{Id} = \sum_{j\geq -1} \Delta_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \tag{2.5}$$

and thus extend the definition for dyadic blocks to tempered distribution.

Proposition 2.2.3. *Let u be in $\mathcal{S}'(\mathbb{R}^n)$. If (χ, ϕ) is a dyadic partition of unity, then $u = \lim_{j\to\infty} S_j u$ in $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. Notice that $\langle u - S_j u, \phi\rangle = \langle u, \phi - S_j\phi\rangle$ for all ϕ in $\mathcal{S}(\mathbb{R}^n)$ and u in $\mathcal{S}'(\mathbb{R}^n)$. For every $|\alpha| \leq k$, we can apply the Leibniz rule to see that

$$\partial^\alpha\Big(\mathcal{F}\phi - \chi(2^{-j}\xi)\mathcal{F}\phi\Big) = \Big(1 - \chi(2^{-j}\xi)\Big)\partial^\alpha\mathcal{F}\phi + \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} 2^{-j|\beta|}(\partial^\beta\chi)(2^{-j}\xi)\partial^{\alpha-\beta}\mathcal{F}\phi.$$

Since $\|\chi\|_{L^\infty} \leq 1$ with $\chi = 1$ near the origin, it follows that

$$\|\mathcal{F}(\phi - S_j\phi)\|_{k,\mathcal{S}} \leq C_k \|\mathcal{F}\phi\|_{k,\mathcal{S}}$$

for j large enough. Because the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^n)$ the claim is proved.. □

When defined on $S'(\mathbb{R}^n)$, we refer to (2.5) as the *Littlewood-Paley decomposition* or simply dyadic decomposition of the tempered distribution u.

We now consider two extremely useful Bernstein-type estimates.

Lemma 2.2.4. *Let $\mathcal{C}$ be an annulus and B a ball. A constant C exists such that for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $q \geq p$, $\lambda > 0$ and any function u of L^p, we have*

$$\text{Supp } \hat{u} \subset \lambda B \Longrightarrow \|D^k u\|_{L^q} \overset{def}{=} \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{(k+n)+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}$$
$$\text{Supp } \hat{u} \subset \lambda \mathcal{C} \Longrightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\phi = 1$ near B and $|\alpha| \leq k$. Then we have $\hat{u}(\xi) = \hat{u}(\xi)\phi(\lambda^{-1}\xi)$ and therefore

$$\partial^\alpha u = \lambda^{|\alpha|-n} u * \partial^\alpha g(\lambda x) \quad \text{with} \quad \partial^\alpha g \overset{\text{def}}{=} \partial^\alpha \mathcal{F}^{-1}\phi.$$

Applying Young's convolution inequality we get

$$\|\partial^\alpha u\|_{L^q} \leq \lambda^{|\alpha|-n} \|\partial^\alpha g(\lambda\cdot)\|_{L^r} \|u\|_{L^p} \quad \text{with} \quad \frac{1}{r} = -\frac{1}{p} + \frac{1}{q} + 1.$$

Since

$$\|\partial^\alpha g(\lambda\cdot)\|_{L^r} \leq \lambda^{(-1+\frac{1}{p}-\frac{1}{q})n} \|\partial^\alpha g\|_{L^r} \quad \text{and} \quad \|\partial^\alpha g\|_{L^r} \leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \leq C^{k+1}$$

we therefore conclude our first claim. The right hand side of the second claim follows readily from the first one by putting $q = p$. On the other hand, using the algebraic identity that for some constants A_α,

$$|\xi|^{2k} = \sum_{1 \leq j_1, \ldots, j_k \leq n} \xi_{j_1}^2 \cdots \xi_{j_k}^2 = \sum_{|\alpha|=k} A_\alpha (i\xi)^\alpha (-i\xi)^\alpha,$$

we have for every $\phi \in \mathcal{D}(\mathbb{R}^n \backslash \{0\})$ such that $\phi = 1$ near $\mathcal{C}$, that

$$u = \sum_{|\alpha|=k} g_{\alpha,\lambda} * \partial^\alpha u \quad \text{with} \quad g_{\alpha,\lambda} \overset{\text{def}}{=} A_\alpha (-i\xi)^\alpha |\xi|^{-2k} \phi(\lambda^{-1}\xi).$$

Thus, applying again Young's inequality

$$\|u\|_{L^p} \leq \sum_{|\alpha|=k} \|g_{\alpha,\lambda}\|_{L^1} \|\partial^\alpha u\|_{L^p},$$

and it suffices to note that

$$\|g_{\alpha,\lambda}\|_{L^1} = \lambda^{|\alpha|} \|g_\alpha\|_{L^1}, \quad \text{with} \quad g_\alpha \overset{\text{def}}{=} g_{\alpha,1}$$

to conclude the proof. □

2.3 Hölder-Besov Spaces

We are now ready to discuss the function spaces in which our analysis will take place.

Definition 2.3.1. Let $s \in \mathbb{R}$, the *Hölder-Besov spaces* $\mathcal{C}^s(\mathbb{R}^n; \mathbb{C})$ consists of all tempered distributions u such that

$$\|u\|_{\mathcal{C}^s} \stackrel{\text{def}}{=} \left\| (2^{js} \|\Delta_j u\|_{L^\infty})_{j\in\mathbb{Z}} \right\|_{\ell^\infty(\mathbb{Z}^n)} < \infty.$$

Clearly this defines a norm.

In the above definition we fixed a particular dyadic partition of unity. In what follows we will often assume, without further explanation that such a choice coincide with the one constructed in (2.2.1). However, it is important to know that the definition of Besov space is independent of such choices. For this we need the following important lemma.

Lemma 2.3.2. *Let $\mathcal{C}'$ be an annulus and B a ball. Let $(u_j)_{j\geq -1}$ be a sequence of smooth function on $\mathbb{R}^n$ and*

1. *there exists constant C' such that the functions satisfy*
$$\text{Supp } \hat{u}_j \subset 2^j\mathcal{C}' \quad \text{and} \quad \|u_j\|_{L^\infty} \leq C' 2^{-js}, \; j \geq -1, \; s \in \mathbb{R}$$
or;

2. *there exists constant C such that the functions satisfy*
$$\text{Supp } \hat{u}_j \subset 2^j B \quad \text{and} \quad \|u_j\|_{L^\infty} \leq C 2^{-js}, \; j \geq -1, \; s > 0$$

In either case, we then have

$$u \stackrel{\text{def}}{=} \sum_{j\geq -1} u_j \in \mathcal{C}^s \quad \text{and} \quad \|u\|_{\mathcal{C}^s} \leq C_s \left\| (2^{js} \|u_j\|_{L^\infty})_{j\geq -1} \right\|_{\ell^\infty(\mathbb{Z})}.$$

Proof. If the Fourier transform of u_j is supported in $2^j\mathcal{C}'$. then there exists positive integer N such that

$$|j' - j| > N \Longrightarrow \Delta_{j'} u_j = 0.$$

Thus we have

$$\|\Delta_j u\|_{L^\infty} \leq \sum_{|j'-j|\leq N} \|\Delta_j u_{j'}\|_{L^\infty} \leq 2^{-js} C_s \left\| (2^{js} \|u_j\|_{L^\infty})_{j\geq -1} \right\|_{\ell^\infty(\mathbb{Z})}. \tag{2.6}$$

If the Fourier transform of u_j is supported in $2^j B$. Then there exists positive integer N' such that

$$j' > j + N \Longrightarrow \Delta_{j'} u_j = 0,$$

and the same bound (2.6) applies with a geometric series. The claim follows. □

This gives rises to the aforementioned corollary.

Corollary 2.3.3. *The space $\mathcal{C}^s$ does not depend on the choice of dyadic decomposition.*

Proof. Let u be in $\mathcal{C}^s$ and suppose (χ', φ') is another dyadic partition of unity with the corresponding dyadic blocks Δ'_j. Proposition (2.2.2), (2.2.3) together with Lemma (2.3.2) then implies there exists constant C_s such that

$$u = \sum_{j\geq -1} \Delta'_j u \quad \text{and} \quad \|u\|_{\mathcal{C}^s} \leq C_s \left\| (2^{js} \|\Delta'_j u\|_{L^\infty})_{j\in\mathbb{Z}} \right\|_{\ell^\infty(\mathbb{Z})}.$$

Hence any two $\mathcal{C}^s$ norms must be equivalent. □

2.3.1 Completeness and Embeddings

Let us gather some topological properties of Hölder-Besov spaces. A natural first observation is that they are complete.

Proposition 2.3.4. *For every $s \in \mathbb{R}$ the space $\mathcal{C}^s$ is continuously embedded into $\mathcal{S}'$. Moreover, $(\mathcal{C}^s, \|\cdot\|_{\mathcal{C}^s})$ is a Banach space.*

Proof. Let us apply the idea used in the proof of Lemma (2.2.4). Since the Fourier transform of $\Delta_j u$ is supported in $2^j\mathcal{C}$, we can choose $\phi \in \mathcal{D}(\mathcal{C}\backslash\{0\})$ such that $\phi = 1$ near $\mathcal{C}$, then for all $j \geq 0$,

$$\Delta_j u = 2^{-jk} \sum_{|\alpha|=k} 2^{jn} g_\alpha(2^j \cdot) * \partial^\alpha \Delta_j u.$$

Hence, for any Schwartz function $\phi \in \mathcal{S}$, we have

$$\langle \Delta_j u, \phi \rangle = (-1)^k 2^{-js} \sum_{|\alpha|=k} \langle \Delta_j u, \check{g}_\alpha * \partial^\alpha \phi \rangle,$$

where we recall that

$$g_\alpha = A_\alpha(-i\xi)^\alpha |\xi|^{-2k} \phi(\xi)$$

as in the proof of the Bernstein estimates Lemma 2.2.4. Hence we can find constant M_k such that

$$|\langle \Delta_j u, \phi \rangle| \leq 2^{-j}(2^{j(1-k)} \|\Delta_j u\|_{L^\infty}) \|\phi\|_{M_k,\mathcal{S}} \sum_{|\alpha|=k} \|\check{g}_\alpha\|_{L^{\frac{1}{2}}}.$$

Since we can choose k toe be as large as we want, it follows that

$$|\langle \Delta_j u, \phi \rangle| \leq C_k 2^{-j} \|u\|_{\mathcal{C}^s} \|\phi\|_{M_{k,\mathcal{S}}}.$$

Now take summation over all $j \geq -1$ to see that

$$|\langle u, \phi \rangle| \leq C_k \|u\|_{\mathcal{C}^s} \|\phi\|_{M_k,\mathcal{S}}.$$

This concludes the continuity of the embedding $\mathcal{C}^s \hookrightarrow \mathcal{S}'$. If $(u^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^s$, then $(\langle u^{(n)}, \phi \rangle)_{k\in\mathbb{N}}$ must be a Cauchy sequence in $\mathbb{R}$. Thus by the Banach-Steinhaus theorem in Fréchet space, the limit $\lim_{k\to\infty}\langle u^{(k)}, \phi \rangle$ define a tempered distribution. Moreover, $(\Delta_j u^{(k)})_{k\in\mathbb{N}}$ needs to be a Cauchy sequence in L^∞ for every j. Since $L^\infty \hookrightarrow \mathcal{S}'$ is a continuous embedding and that the actions of dyadic blocks are continuous in $\mathcal{S}'$ we also have $\lim_{k\to\infty} \Delta_j^{(k)} u^{(k)} = \Delta_j u$. Therefore, for every $\epsilon > 0$ there exists a positive integer N such the

$$2^{js} \|u^{(k)} - u^{(k')}\|_{\mathcal{C}^s} \leq \|u^{(k)} - u^{(k')}\|_{\mathcal{C}^s} \leq \epsilon$$

for all $k, k' \geq N$. By taking the limit as k' goes to infinity and supremum over all j we obtain the desired inequality. □

Let us now state a few more useful embedding properties.

Proposition 2.3.5. *Let s be a real number, then the Hölder-Besov spaces $\mathcal{C}^s$ satisfies the following embedding properties:*

1. $\mathcal{C}^s \hookrightarrow L^\infty$ *if* $s > 0$*;*

2. $L^\infty \hookrightarrow \mathcal{C}^s$ *if* $s \leq 0$*;*

3. $\mathcal{C}^s \hookrightarrow \mathcal{C}^r$ *if* $r \leq s$*;*

4. *Let* $(s, r) \in [1, \infty]^2$ *and* $B^s_{p,r}$ *to be the space of all tempered distributions such that*

$$\|u\|_{B^s_{p,r}} \overset{def}{=} \left\|(2^{js}\|\Delta_j u\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}.$$

Let $1 \leq p_1 \leq p_2 \leq \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$*, then we have* $B^s_{p_1,r_2} \hookrightarrow B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.

Proof. The first embedding is a strightforward consequences of

$$\|u\|_{L^\infty} \leq \sum_{j\geq -1} \|\Delta_j u\|_{L^\infty} \leq \sum_{j\geq -1} 2^{-js}\|u\|_{\mathcal{C}^s}$$

and the fact that $s > 0$. For the second claim we have

$$\|u\|_{\mathcal{C}^s} = \left\|(2^{-j|s|}\|\Delta_j u\|_{L^\infty})_{j\in\mathbb{Z}}\right\|_{\ell^\infty(\mathbb{Z})} \leq \left\|(\|\Delta_j u\|_{L^\infty})_{j\in\mathbb{Z}}\right\|_{\ell^\infty(\mathbb{Z})} \leq C\|u\|_{L^\infty}$$

whenever $j \geq 0$ and $s \leq 0$. The third claim is a trivial consequence of the first two. Using the embeddings $\ell^{r_1}(\mathbb{Z}) \hookrightarrow \ell^{r_2}(\mathbb{Z})$ and an application of the Bernstein lemma which yields

$$\|\Delta_j u\|_{L^{p_2}} \leq C2^{jn(\frac{1}{p_1}-\frac{1}{p_2})}\|\Delta_j u\|_{L^{p_1}},$$

we arrive at the final claim as well. This concludes the proof. □

2.3.2 The Hölder Condition

In this subsection we will justify why the spaces $\mathcal{C}^s$ are called Hölder-Besov spaces. Indeed, let s be in $(0, 1)$, recall that the space of *Hölder continuous functions* consists of real valued functions u such that $|u(x)-u(y)| \leq C|x-y|^s$ for some positive constant $C > 0$. If s is instead chosen to be in $(0, \infty)$, then a natural generalisation of this idea is to consider the space $C^{s,s-[s]}_b$ of all bounded real valued functions such that

$$\|u\|_{C^{s,s-[s]}_b} \overset{\text{def}}{=} \|u\|_{C^{[s]}_b} + \max_{|\alpha|=[s]} \sup_{x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{s-[s]}} < \infty.$$

It turns out that if s is not an integer, then $C^{s-[s]}_b$ could be characterised by $\mathcal{C}^s$.

Proposition 2.3.6. *Let s be in $\mathbb{R}^+\backslash\mathbb{N}$, then the spaces $\mathcal{C}^s$ and $C^{[s],s-[s]}_b$ coincide.*

As a non-standard result, let us first prove a Taylor-type estimate for elements of $C^{[s],s-[s]}_b$.

Lemma 2.3.7. *Let s be in $\mathbb{R}^+\backslash\mathbb{N}$. Then for every $u \in C_b^{[s],s-[s]}$ there exists positive constant C_s such that*

$$\Big|u(x) - \sum_{|\alpha|\leq[s]} \frac{(x-y)^\alpha}{\alpha!}\partial^\alpha u(y)\Big| \leq \frac{C_s}{[s]!}|x-y|^s.$$

Proof. For functions $f : \mathbb{R} \to \mathbb{R}$ and $\theta' \in (0,1)$, it is easy to see the expansion

$$f(1) = \sum_{k\leq[s]} \frac{\theta^k}{k!}\frac{d^k}{d\theta^k}f(0) + \frac{\theta^{[s]}}{[s]!}\Big(\frac{d^{[s]}}{d\theta^{[s]}}f(\theta') - \frac{d^{[s]}}{d\theta^{[s]}}f(0)\Big).$$

Apply the above identity to $f(\theta) = u\big(y + \theta(x-y)\big)$, we obtain the expression

$$u(x) = \sum_{|\alpha|\leq[s]} \frac{(x-y)^\alpha}{\alpha!}\partial^\alpha u(y) + \sum_{|\alpha|=[s]} \frac{(x-y)^\alpha}{\alpha!}\Big(\partial^\alpha u\big(y+\theta'(x-y)\big) - \partial^\alpha u(y)\Big), \quad \text{with}$$
$$\Big|\sum_{|\alpha|=[s]} \frac{(x-y)^\alpha}{\alpha!}\Big(\partial^\alpha u\big(y+\theta'(x-y)\big)\Big) - \partial^\alpha u(y)\Big| \leq \frac{C_s}{[s]!}|x-y|^s$$

which is the desired result. □

From the classical estimate $\|\partial^\alpha u\|_{C_b^{s,s-[s]}} \leq C_s\|u\|_{C_b^{s+|\alpha|,s-[s]}}$, it is also natural for the differential operator defined on $\mathcal{C}^s$ spaces to be continuous.

Lemma 2.3.8. *Let (s,α) be in $\mathbb{R}\times\mathbb{N}^n$. Then the differential operator $\partial^\alpha : \mathcal{C}^s \to \mathcal{C}^{s-|\alpha|}$ is bounded.*

Proof. Since for each j we have

$$\partial^\alpha\Delta_j u = \partial^\alpha\mathcal{F}^{-1}\Big(\varphi(2^{-j}\xi)\hat{u}(\xi)\Big) = \mathcal{F}^{-1}\Big((i\xi)^\alpha\varphi(2^{-j}\xi)\hat{u}(\xi)\Big) = \mathcal{F}^{-1}\Big(\varphi(2^{-j}\xi)\widehat{\partial^\alpha u}(\xi)\Big).$$

By an application of the Bernstein Lemma (2.2.4), there exists annulus $\mathcal{C}$ and constant C such that

$$2^{j(s-|\alpha|)}\|\partial^\alpha\Delta_j u\|_{L^\infty} \leq C^2 2^{js}\|\Delta_j u\|_{L^\infty}.$$

Summing over all j yields the required estimate. □

We are now ready to prove Proposition (2.3.6).

Proof of Proposition 2.3.6. Let us note first that, since

$$\int_{\mathbb{R}^n} x^\alpha h(x)dx = i^{-\alpha}\int_{\mathbb{R}} \mathcal{F}^{-1}\partial^\alpha\varphi dx = 0, \quad \forall\alpha\in\mathbb{N}^n,$$

owing to the binomial theorem, we can write

$$\Delta_j u(x) = 2^{jn}\int_{\mathbb{R}^n} h\big(2^j(x-y)\big)\Big(u(y) - \sum_{|\alpha|=1}^{[s]} \frac{(y-x)^\alpha}{\alpha!}\partial^\alpha u(x)\Big)dy.$$

So by an application of the previously established Taylor expansion, we have

$$\|\Delta_j u\|_{L^\infty} \leq \frac{(1+C_s)}{[s]!}\|u\|_{L^\infty} 2^{-js}\|h|\cdot|^s\|_{L^1},$$

together with the obvious bound $\|\Delta_{-1}u\|_{L^\infty} \leq C\|u\|_{L^\infty}$ it is easy to see that

$$\|u\|_{\mathcal{C}^s} \leq \frac{(1+C_s)}{[s]!}\|u\|_{C_b^{[s],s-[s]}}\|h|\cdot|^s\|_{L^1}.$$

In virtue of the continuity of the differential operator, it suffices to prove the estimate $\|\partial^\alpha u\|_{C_b^{0,s-[s]}} \leq C_s\|u\|_{\mathcal{C}^s}$. Suppose now that $|x-y|>1$, then by definition

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq 2\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}}|x-y|^{s-[s]},$$

we already have one part of the claim. If instead $|x-y|\leq 1$, then choose j' such that there exists c', C' which satisfies $c'2^{-j'} \leq |x-y| \leq C'2^{-j'}$. We have

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq \sum_{j<j'}|\partial^\alpha\Delta_j u(x) - \partial^\alpha\Delta_j u(y)| + \sum_{j\geq j'}|\partial^\alpha\Delta_j u(x) - \partial^\alpha\Delta_j u(y)|.$$

For $j<j'$, we make an application of the Bernstein lemma and the mean value theorem to obtain that

$$\begin{aligned}|\partial^\alpha\Delta_j u(x) - \partial^\alpha\Delta_j u(y)| &\leq C^2 2^{j(1-s+[s])}2^{j(s-[s])}\|\partial^\alpha\Delta_j u\|_{L^\infty}|x-y| \\ &\leq C^2 2^{j(1-s+[s])}\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}}|x-y|.\end{aligned}$$

For $j\geq j'$, we simply have

$$|\partial^\alpha\Delta_j u(x) - \partial^\alpha\Delta_j u(y)| \leq 2\|\partial^\alpha\Delta_j u\|_{L^\infty} \leq 2^{j([s]-s)}\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}}.$$

Putting everything together we see that

$$\begin{aligned}|\partial^\alpha u(x) - \partial^\alpha u(y)| &\leq C^2 2^{j'(1-s+[s])}\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}}|x-y| + 2^{j'([s]-s)}\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}} \\ &\leq C^2|x-y|^{s-[s]}\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}} + |x-y|^{s-[s]}\|\partial^\alpha u\|_{\mathcal{C}^{s-[s]}}\end{aligned}$$

which is the desired inequality. This concludes the proof of the claim. □

2.4 Paradifferential Calculus

A central theme of our approach is to develop a theory of integration for distributions driven against irregular noise. For this purpose it is first necessary to make sense of the products between distributions. One way to do this is to consider paraproduct.

2.4.1 The Bony Decomposition

Consider two tempered distributions u and v. It is natural to study simple operations between them. In particular, since we would like distribution to generalise the space of functions, it would be useful to have a notion of addition and multiplication. Addition is easily understood. However, it is generally difficult to make sense of the product

uv, and paraproduct is the mathematical tool of doing that via duality. We have already developed a representation of u and v that involves duality, namely, the dyadic decomposition

$$u = \sum_{j\geq -1} \Delta_j u \quad \text{and} \quad v = \sum_{j'\geq -1} \Delta_{j'} v$$

which are defined using Fourier transform. Formally, we could write

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v \stackrel{\text{def}}{=} \sum_{j\geq -1} S_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) \stackrel{\text{def}}{=} \sum_{|k-j|\leq 1} \Delta_k u \Delta_j v.$$

Constructed this way, we say that $T_u v$ is the *paraproduct of v by u*, and $R(u, v)$ is the *remainder of u and v*. Such a decomposition is called the *Bony decomposition*. The main continuity properties of the paraproduct and the remainder are described below.

Proposition 2.4.1. *For any couple of real numbers (s, r) we have*

$$T : L^\infty \times \mathcal{C}^s \to \mathcal{C}^s : \ (u, v) \longmapsto \sum_{j\geq -1} S_{j-1} u \Delta_j v, \ s \in \mathbb{R},$$
$$T : \mathcal{C}^s \times \mathcal{C}^r \to \mathcal{C}^{s+r} : \ (u, v) \longmapsto \sum_{j\geq -1} S_{j-1} \Delta_j v, \ s < 0, \ r \in \mathbb{R},$$
$$R : \mathcal{C}^s \times \mathcal{C}^r \to \mathcal{C}^{s+r} : \ (u, v) \longmapsto \sum_{|k-j|\leq 1} \Delta_k u \Delta_j v, \ s + r > 0,$$

are bounded bilinear operators.

Proof. We remark that for $j, j' \geq 0$

$$\Delta_{j'} u \Delta_j v = \mathcal{F}^{-1}\Big(\varphi(2^{-j'} \hat{u}(\xi)) * \varphi(2^{-j}\xi)\hat{v}(\xi)\Big)$$

and for $j' = -1$,

$$\Delta_{j'} u \Delta_j v = \mathcal{F}^{-1}\Big(\chi(2^{-j'} \hat{u}(\xi)) * \varphi(2^{-j}\xi)\hat{v}(\xi)\Big).$$

Assume u is in L^∞ and v is in $\mathcal{C}^s$. Then by the above identities $\mathcal{F}(S_{j-1} u \Delta_j v)$ is supported in $2^j \tilde{\mathcal{C}}$. Since

$$\|S_{j-1} u \Delta_j v\|_{L^\infty} \leq \|S_{j-1} u\|_{L^\infty} \|\Delta_j v\|_{L^\infty} \leq C 2^{-js} \|u\|_{L^\infty} \|v\|_{\mathcal{C}^s}.$$

By Lemma 2.3.2 the first bound follows. If we have $(u, v) \in \mathcal{C}^s \times \mathcal{C}^r$ with $r < 0$. Then the second bound follows from

$$\|S_{j-1} v \Delta_j u\|_{L^\infty} \leq C_r 2^{(-j+1)r} \|\Delta_j v\|_{L^\infty} \|v\|_{\mathcal{C}^r} \leq C_{s,r} 2^{-j(r+s)} \|u\|_{\mathcal{C}^s} \|v\|_{\mathcal{C}^r}$$

If $s + r > 0$, then write

$$R(u, v) = \sum_j R_j(u, v) \quad \text{with} \quad R_j(u, v) \stackrel{\text{def}}{=} \sum_{|\nu|\leq 1} \Delta_{j-\nu} u \Delta_j v.$$

Notice there exists a ball B' such that $\mathcal{F}R_j$ is uniformly supported in $2^j B'$ and

$$\|R_j(u, v)\|_{L^\infty} \leq \sum_{|\nu|\leq 1} 2^{(j-\nu)s} \|\Delta_{j-\nu} u\|_{L^\infty} 2^{jr} \|\Delta_j v\|_{L^\infty} 2^{(-j+\nu)s} 2^{-jr}$$
$$\leq 2^{-j(s+r)} \|u\|_{\mathcal{C}^s} \|v\|_{\mathcal{C}^r} \sum_{|\nu|\leq 1} 2^{-\nu s} < \infty.$$

By the second part of 2.3.2 this proves the claim. $\square$

2.4.2 Actions on Besov Spaces and Paralinearisation

In this section we will study various actions on the Hölder-Besov spaces and highlight a number of important calculations. Our main result will concern the effects of left composition by sufficiently smooth functions.

Proposition 2.4.2. *Let s be a positive real number and σ be the smallest positive integer such that $\sigma \geq s$. If u be long to $\mathcal{C}^s$, then so does $f \circ u$ for every $f \in C_b^\sigma$, and we have*

$$\|f \circ u\|_{\mathcal{C}^s} \leq C(s, \|u\|_{L^\infty})\|f\|_{C_b^\sigma}\|u\|_{\mathcal{C}^s}$$

for some constant $C(s, \|u\|_{L^\infty})$ depending on s and $\|u\|_{L^\infty}$.

Proof. We introduce formally the series

$$\sum_{j\geq -2} f_j \quad \text{with} \quad f_j \stackrel{\text{def}}{=} f(S_{j+1}u) - f(S_j u) \quad \text{and} \quad f_{-2} \stackrel{\text{def}}{=} f_{|x=0}.$$

For $j \geq -1$, it is clear that

$$f_j = m_j \Delta_j u \quad \text{with} \quad m_j \stackrel{\text{def}}{=} \int_0^1 f'(S_j u + t'\Delta_j u)dt'. \tag{2.7}$$

We will need to make use of the following lemma.

Lemma 2.4.3. *Let g be a function from $\mathbb{R}^2$ to $\mathbb{R}$ such that $g \in C_b^\sigma$. For $j \geq -1$, define*

$$m_j(g) \stackrel{def}{=} g(S_j u, \Delta_j u).$$

For any bounded function u, there exists positive constant C_α such that

$$\forall \alpha \in \mathbb{N}^n, \quad \|\partial^\alpha m_j(g)\|_{L^\infty} \leq C_\alpha(1 + \|u\|_{L^\infty})^{|\alpha|}\|g\|_{C_b^\alpha} 2^{j|\alpha|}.$$

Proof. By the Faá di Bruno's formula, we have

$$\partial^\alpha m_j(g) = \sum_{p_1,p_2,\nu} C^\nu_{p_1,p_2}\Big(\prod_{1\leq|\beta|\leq|\alpha|}(\partial^\beta S_j u)^{\nu_{\beta_1}}(\partial^\beta \Delta_j u)^{\nu_{\beta_2}}\Big)\partial_1^{p_1}\partial_2^{p_2} g(S_j u, \Delta_j u),$$

where the coefficients $C^\nu_{p_1,p_2}$ are nonnegative integers, and the sum is taken over those p_1, p_2 and ν such that $1 \leq p_1 + p_2 \leq |\alpha|$ such that

$$\sum_{1\leq|\beta|\leq|\alpha|} \nu_{\beta_j} = p_1 \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad \sum_{1\leq|\beta|\leq|\alpha|} \beta(\nu_{\beta_1} + \nu\beta_2) = \alpha.$$

Note that there exists a constant C' such that

$$\max\{\|\Delta_j u\|_{L^\infty}, \|S_j u\|_{L^\infty}\} \leq C'\|u\|_{L^\infty}.$$

If g is sufficiently regular, then its derivatives are bounded up to order α on $B(0, C\|u\|_{L^\infty})$. By an application of the Bernstein estimate we thus arrive at

$$\begin{aligned}\|\partial^\alpha m_j(g)\|_{L^\infty} &\leq \max_{p_1,p_2,\nu} C^\nu_{p_1,p_2}\sum_\nu \prod_{1\leq|\beta|\leq|\alpha|} C^{2|\beta|+3}2^{j|\beta|(\nu_{\beta_1}+\nu_{\beta_2})}\|u\|_{L^\infty}^{\nu_{\beta_1}+\nu_{\beta_2}}\|g\|_{C_b^\alpha}\\ &\leq C^{2|\alpha|+3}_{\nu,p_1,p_2}(1 + \|u\|_{L^\infty})^{|\alpha|}\|g\|_{C_b^\alpha}2^{j|\alpha|}\end{aligned}$$

which is the required estimate. □

Let us now resume the proof of Proposition 2.4.2. It is clear that f_{-2} is in $\mathcal{C}^s$. For $j' \geq -1$ we make the decomposition

$$\Delta_{j'} \sum_{j\geq -1} f_j = \sum_{j<j'} \Delta_{j'}(m_j \Delta_j u) + \sum_{j\geq j'} \Delta_{j'}(m_j \Delta_j u).$$

For the first sum, we take advantage of Lemma 2.4.3 to get that

$$\|\partial^\alpha m_j\|_{L^\infty} \leq \int_0^1 \|\partial^\alpha f'(S_j u + t'\Delta_j u)\|_{L^\infty} dt' \leq C_\alpha (1+\|u\|_{L^\infty})^\sigma \|f'\|_{C_b^\sigma} 2^{j\sigma}$$

whenever $|\alpha| = \sigma$. It then follows from the Bernstein lemma 2.2.4 and the Leibniz rule that

$$\begin{aligned}\|\Delta_{j'}\big(m_j(u)\Delta_j u\big)\|_{L^\infty} &\leq C^{\sigma+1} 2^{-j'\sigma} \|D^\sigma \Delta_{j'}\big(m_j \Delta_j u\big)\|_{L^\infty} \\ &\leq C_\sigma^{2\sigma+2} 2^{-j'\sigma} \sum_{\beta\leq\sigma} C_{\alpha,\beta} 2^{j|\beta|} (1+\|u\|_{L^\infty})^\sigma \|f'\|_{C_b^\sigma} 2^{j(|\alpha|-|\beta|)} \|\Delta_j u\|_{L^\infty} \\ &\leq C'_\sigma (1+\|u\|_{L^\infty})^\sigma \|f'\|_{C_b^\sigma} 2^{(j'-j)(s-\sigma)} (2^{js}\|\Delta_j u\|_{L^\infty}).\end{aligned}$$

Therefore, as

$$\Big\|\Delta_{j'} \sum_{j<j'} (m_j \Delta_j u)\Big\|_{L^\infty} \leq C'_\sigma (1+\|u\|_{L^\infty})^\sigma \|f'\|_{C_b^\sigma} \sum_{j\in\mathbb{Z}} 2^{(j'-j)(s-\sigma)} (2^{js}\|\Delta_j u\|_{L^\infty}),$$

an application of Young's inequality yields

$$\begin{aligned}&\bigg\|\bigg(2^{j's}\Big\|\Delta_{j'} \sum_{j'\geq -1} (m_j \Delta_j u)\Big\|_{L^\infty}\bigg)_{j'\in\mathbb{Z}}\bigg\|_{\ell^\infty(\mathbb{Z})} \\ &\qquad\leq C'_\sigma (1+\|u\|_{L^\infty})^\sigma \|f'\|_{C_b^\sigma} \|(2^{j(s-\sigma)})_{j\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})} \|u\|_{\mathcal{C}^s}.\end{aligned}$$

Likewise, estimation for the second sum follows easily from

$$2^{j's}\|\Delta_{j'}(m_j\Delta_j u)\|_{L^\infty} \leq 2^{(j'-j)s}\|f'\|_{L^\infty} 2^{js}\|\Delta_j u\|_{L^\infty}) \leq \|f'\|_{L^\infty}\|u\|_{\mathcal{C}^s}.$$

This proves the claim. □

Similarly, if f is sufficiently regular, then the action of f on u can be linearised using paraproduct. Such operation is called *paralinearisation*.

Proposition 2.4.4. *Let s be in $(0,1)$ and $r \in (0,s]$. Suppose that f is in $C_b^{1,r/s}$, then there exists a locally bounded operator*

$$R_f : \mathcal{C}^s \to \mathcal{C}^{s+r} : u \longmapsto f\circ u - T_{f'\circ u} u$$

in the sense that

$$\|R_f u\|_{\mathcal{C}^{s+r}} \leq C\|f\|_{C_b^{1,r/s}} (1+\|u\|_{\mathcal{C}^s}^{1+r/s}).$$

If instead f is in $C_b^{2,r/s}$, then R_f is locally Lipschitz continuous in the sense that

$$\|R_f u - R_f v\|_{\mathcal{C}^{s+r}} \leq C\|f\|_{C_b^{2,r/s}} (1+\|u\|_{\mathcal{C}^s}+\|v\|_{\mathcal{C}^s})^{1+r/s}\|u-v\|_{\mathcal{C}^s}.$$

Proof. By dyadic decomposition, we have

$$f \circ u - T_{f' \circ u} u = f \circ u - \sum_{j \geq -1} S_{j-1}(f' \circ u) \Delta_j u = \sum_{j \geq -1} \Delta_j (f \circ u) - S_{j-1}(f' \circ u) \Delta_j u.$$

By construction, the Fourier transform of $\Delta_j(f \circ u) - S_{j-1}(f' \circ u)\Delta_j u$ is supported in a ball of radius larger than $2^j \mathcal{C}$. If $j < 1$ then we easily estimate

$$|\Delta_j(f \circ u) - S_{j-1}(f' \circ u)\Delta_j u| \leq \|f\|_{L^\infty} + \|f'\|_{L^\infty} \|u\|_{L^\infty} \leq C_s \|f\|_{C_b^{1,s/r}} (1 + \|u\|_{\mathcal{C}^s}^{1+s/r}).$$

Fix $j \geq 1$. Since

$$2^{j'n} \int_{\mathbb{R}^n} h\Big(2^{j'}(x-z)\Big) dz = \begin{cases} \varphi(2^{-j'}0) = 0 \text{ if } j' \geq 0, \\ \quad \chi(0) \quad = 1 \text{ if } j' = -1. \end{cases}$$

Set

$$S^{(j)}(x) \stackrel{\text{def}}{=} \sum_{0 \leq j' \leq j-2} 2^{j'n} h(2^{j'} x) + \tilde{h}(x).$$

It suffices to write the Fourier multipliers in convolution to see that

$$\begin{aligned} &\Delta_j(f \circ u) - S_{j-1}(f' \circ u)\Delta_j u \\ &\quad = 2^{jn} \int_{\mathbb{R}^n} h\Big(2^j(x-y)\Big)(f \circ u)(y) dy \\ &\qquad\qquad - 2^{jn} \int_{\mathbb{R}^n \times \mathbb{R}^n} S^{(j)}(x-z)\Big) h\Big(2^j(x-y)\Big)(f' \circ u)(z) u(y) dy dz \\ &\quad = \int_{\mathbb{R}^{2n}} h\Big(2^j(x-y)\Big) 2^{jn} S^{(j)}(x-z)\Big((f \circ u)(y) - (f' \circ u)(z) u(y)\Big) dy dz. \end{aligned}$$

Then by adding the term

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} h\Big(2^j(x-y)\Big) 2^{jn} S^{(j)}(x-z)\Big)\Big(-(f \circ u)(z) + (f' \circ u)(z) u(z)\Big) dy dz \\ &\qquad = \varphi(2^{-j}0) \int_{\mathbb{R}^n} S^{(j)}(x-z)\Big(-(f \circ u)(z) + (f' \circ u)(z) u(z)\Big) dz = 0, \end{aligned}$$

we get

$$\begin{aligned} \Delta_j(f \circ u) - S_{j-1}(f' \circ u)\Delta_j u = &\int_{\mathbb{R}^{2n}} h\Big(2^j(x-y)\Big) 2^{jn} S^{(j)}(x-z) \\ &\times \Big((f \circ u)(y) - (f \circ u)(z) - (f' \circ u)(z)\Big(u(y) - u(z)\Big)\Big) dy dz. \end{aligned}$$

By Proposition 2.3.6 and an application of the Taylor formula of f at $u(y)$, we have

$$|(f \circ u)(y) - (f \circ u)(z) - (f' \circ u)(z)\Big(u(y) - u(z)\Big)| \leq C_s \|f\|_{C_b^{1,r/s}} \|u\|_{\mathcal{C}^s}^{1+r/s} |y-z|^{s+r}.$$

Hence, using the identity

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} |h\Big(2^j(x-y)\Big) 2^{jn} S^{(j)}(x-z)||z-y|^{s+r} dy dz \\ &\qquad\qquad = 2^{-j(s+r)} \int_{\mathbb{R}^{2n}} |h(y) 2^{-jn} S^{(j)}(2^{-j} z)|z-y|^{s+r}| dy dz, \end{aligned}$$

we arrive at

$$|\Delta_j(f \circ u) - S_{j-1}(f' \circ u)\Delta_j u| \leq C_{s,r} 2^{-j(s+r)} \|f\|_{C_b^{1,r/s}} \|u\|_{\mathcal{C}^s}^{1+r/s}.$$

and so the estimate for $R_F u$ follows from Lemma 2.3.2. The estimate for $R_F u - R_F v$ follows from the same estimate. □

For the purpose of treating special additive structures, it is also useful to have the following modification.

Proposition 2.4.5. *Let s be in $(1/3,1/2)$ and $f\in C_b^3$. Then for every $(u,v)\in\mathcal{C}^s\times\mathcal{C}^{2s}$ there exists constant C_s such that*

$$\|R_f(u+v)\|_{\mathcal{C}^s}\le C_s\|f\|_{C_b^3}(1+\|u\|_{\mathcal{C}^s})(\|u+v\|_{\mathcal{C}^s}+\|v\|_{\mathcal{C}^{2s}}).$$

Proof. Owing to the linearity of the paraproduct, we can write

$$f(u+v)-T_{f'(u+v)}(u+v)=f(u+v)-T_{f'(u+v)}u+T_{f'(u+v)}v.$$

By making a dyadic decomposition, we have

$$f(u+v)-T_{f'(u+v)}u=\sum_{j\ge-1}\Delta_j\Big(f(u+v)\Big)-S_{j-1}\Big(f'(u+v)\Big)\Delta_j u.$$

For $j\ge 0$, proceed as in the proof of Lemma 2.4.4, we can consider the expression

$$\begin{aligned}&\Delta_j\Big(f(u+v)\Big)-S_{j-1}\Big(f'(u+v)\Big)\Delta_j u\\&\quad=\int_{\mathbb{R}^{2n}}2^{jn}h\Big(2^j(x-y)\Big)S^{(j)}(x-z)f\Big(u(y)+v(y)\Big)\\&\qquad-f\Big(u(z)+v(y)\Big)-f'\Big(u(z)+v(z)\Big)(u(y)-u(z))+f\Big(u(z)+v(y)\Big)dydz.\end{aligned}$$

Applying a first order Taylor's formula, we get

$$\begin{aligned}&f\Big(u(y)+v(y)\Big)=f\Big(u(z)+v(y)\Big)+f'\Big(u(z)+u(y)\Big)\Big(u(y)-u(z)\Big)\\&\quad+f'\Big(u(z)+v(y)\Big)\Big(u(y)-u(z)\Big)-f'\Big(u(z)+v(z)\Big)\Big(u(y)-u(z)\Big)+J(y)\Big(u(y)-u(z)\Big)^2.\end{aligned}$$

Applying Proposition 2.3.6, the remainder can be estimated by

$$|J(y)\Big(u(y)-u(z)\Big)^2|\le C\|f\|_{C_b^3}\|u\|_{\mathcal{C}^s}^2|y-z|^{2s}.$$

On the other hand,

$$\begin{aligned}&|f'\Big(u(z)+v(y)\Big)\Big(u(y)-u(z)\Big)-f'\Big(u(z)+v(y)\Big)\Big(u(y)-u(z)\Big)|\\&\qquad\le\|f\|_{C_b^3}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^{2s}}|y-z|^{2s}.\end{aligned}$$

For what's left of the integral, we estimate

$$\begin{aligned}&\Big|\int_{\mathbb{R}^{2n}}2^{jn}h\Big(2^j(x-y)\Big)S^{(j)}(x-z)\Big)f\Big(u(z)+v(y)\Big)dydz\Big)\Big|\\&\qquad\le C_s\int_{\mathbb{R}^n}S^{(j)}(x-z)\sup_{y\in\mathbb{R}^n}|f\Big(u(z)+v(\cdot)\Big)|dz\\&\qquad\le 2^{-2js}C_s\|f\|_{C_b^3}\|v\|_{\mathcal{C}^{2s}}\int_{\mathbb{R}^n}\frac{1}{(1+|z|^2)^N}\sum_{j'}\frac{1}{2^{2j'N}}dz.\end{aligned}$$

For $j=-1$ we simply have

$$\|\Delta_{-1}f(u+v)-S_{-2}\Big(f'(u+v)\Big)\Delta_j u\|_{L^\infty}\le C_s2^s\|f\|_{C_b^3}\|u+v\|_{\mathcal{C}^{2s}}.$$

Since we can easily bound $\|T_{f(u+v)}v\|_{\mathcal{C}^{2s}} \leq C_s\|f\|_{C_b^3}\|v\|_{\mathcal{C}^{2s}}$, by combining all the estimates and using a trick $\|u\|_{\mathcal{C}^s} \leq \|u+v\|_{\mathcal{C}^s} + \|v\|_{\mathcal{C}^s}$ we see that

$$\begin{aligned}
&\|\Delta_j f(u+v) - S_{j-1}\big(f'(u+v)\big)\Delta_j u\|_{L^\infty} \\
&\qquad \leq C_s(1+\|u\|_{\mathcal{C}^s})(\|u+v\|_{\mathcal{C}^s}+\|v\|_{\mathcal{C}^s})\int_{\mathbb{R}^{2n}} 2^{jn}h\Big(2^j(x-y)\Big)S^{(j)}(x-z)|y-z|^{2s}dydz \\
&\qquad\qquad \leq C_s(1+\|u\|_{\mathcal{C}^s})(\|u+v\|_{\mathcal{C}^s}+\|v\|_{\mathcal{C}^s})2^{-2js}.
\end{aligned}$$

By Lemma 2.3.2 the claim follows. □

We conclude this subsection by presenting explicit estimates for two important operations. We have already shown that the the differential operator is continuous between the Hölder-Besov spaces, an equally important operation is multiplication by smooth functions.

Lemma 2.4.6. *Let s be any real number and $\phi \in \mathcal{S}$. Then the multiplication operator $M_\phi u \overset{def}{=} \phi u$ is bounded from $\mathcal{C}^s$ to $\mathcal{C}^s$.*

Proof. The proof relies on Bony's decomposition. Indeed, according to the paraproduct estimates, we have

$$\begin{aligned}
\|T_\phi u\|_{\mathcal{C}^s} &\leq C_s\|\phi\|_{L^\infty}\|u\|_{\mathcal{C}^s}, \\
\|T_u\phi\|_{\mathcal{C}^s} &\leq \begin{cases} C_s\|\phi\|_{\mathcal{C}^s}\|u\|_{L^\infty} \leq C_s\|\phi\|_{\mathcal{C}^s}\|u\|_{\mathcal{C}^s}, & \text{if } s>0, \\ C_s\|\phi\|_{\mathcal{C}^1}\|u\|_{\mathcal{C}^{s-1}} \leq C_s\|\phi\|_{C_b^1}\|u\|_{\mathcal{C}^s} & \text{if } s\leq 0. \end{cases}
\end{aligned}$$

Likewise,

$$\|R(\phi,u)\|_{\mathcal{C}^s} \leq \begin{cases} \|\phi\|_{L^\infty}\|u\|_{\mathcal{C}^s} \text{ if } s>0, \\ \|\phi\|_{1-s}\|u\|_{\mathcal{C}^s} \text{ if } s\leq 0, \end{cases}$$

which completes the proof. □

When solving differential equations we will also make use of the scaling operation. For $\lambda > 0$, the *scaling operator* is the function $\Lambda_\lambda u \overset{\text{def}}{=} u(\lambda\cdot)$.

Lemma 2.4.7. *Let s be any nonzero real number and $\lambda > 0$. Then the scaling operator $\Lambda_\lambda : \mathcal{C}^s \to \mathcal{C}^s$ is bounded with norm $\lambda^s \leq \|\Lambda_\lambda\|_{\mathcal{L}(\mathcal{C}^s)}$*

Proof. First observe the identity

$$\varphi(2^{-\log_2\lambda - j}D)\Lambda_\lambda u = 2^{jn}\int_{\mathbb{R}^n} h(2^{j+\log_2\lambda}x - 2^j y)u(y)dy = \Lambda_\lambda\Delta_j u,$$

and likewise

$$\chi(2^{-\log_2\lambda - j}D)\Lambda_\lambda u = \Lambda_\lambda\Delta_{-1}u.$$

Since there exists constant C_s such that

$$2^{js}\|\varphi(2^{-\log_2\lambda - j}D)\Lambda_\lambda u\|_{L^\infty} \leq C_s\lambda^s 2^{js}\|\Delta_j u\|_{L^\infty} \quad \text{and}$$

$$2^{-s}\|\chi(2^{-\log_2\lambda + 1}D)\Lambda_\lambda u\|_{L^\infty} \leq C_s\lambda^s 2^{-1}\|\Delta_{-1}u\|_{L^\infty},$$

and because any scaling of (χ, φ) also define a partition of unity

$$\Delta_{j'} \Lambda_\lambda u = \sum_{|j-j'|\leq 2} \varphi(2^{-\log_2 \lambda - j} D)\Lambda_\lambda u,$$

we can conclude that

$$2^{j's}\|\Delta_{j'}\Lambda_\lambda u\|_{L^\infty} \leq \sum_{|j-j'|\leq 2} 2^{(j'-j)s}(2^{js}\|\varphi(2^{-\log_2\lambda - j}D)\Lambda_\lambda u\|_{L^\infty}) \leq C_s\|u\|_{\mathcal{C}^s}$$

for j' large enough. The case of lower frequency follows by the same argument. This concludes the proof. □

2.4.3 Commutator Estimates

We now prove several key estimates which will be needed throughout the rest of our discussion. We begin with the following basic estimate.

Lemma 2.4.8. *Let s be in $(0,1)$ and $(u,v) \in \mathcal{C}^s \times L^\infty$. Then there exists positive constant C such that*

$$\|[\Delta_j, u]v\|_{L^\infty} \leq 2^{-js}C\|u\|_{\mathcal{C}^s}\|v\|_{L^\infty}, \quad j \geq -1.$$

Proof. In order to prove this lemma, it suffices to rewrite Δ_j as a convolution operator. Indeed,

$$[\Delta_j, u]v = \Delta_j(uv) - u\Delta_j v = 2^{jn}\int_{\mathbb{R}^n} h\Big(2^j(x-y)\Big)u(y)(v(y)-v(x))dy, \quad h = \mathcal{F}^{-1}\varphi.$$

By identifying $\mathcal{C}^s$ with the space of Hölder continuous function $C^{0,s}$, for almost every y and all $x \in \mathbb{R}^n$ such that $x - y \neq 0$, we have

$$\begin{aligned}|[\Delta_j, u]v| \leq 2^{jn}\|u\|_{L^\infty}\int_{\mathbb{R}^n} |h\Big(2^j(x-y)\Big)(x-y)^s|\frac{|v(y)-v(x)|}{|y-x|^s}dy \\ \leq 2^{jn}\|u\|_{L^\infty}\|v\|_{\mathcal{C}^s}C\int_{\mathbb{R}^n}|h(2^jy)y^s|dy.\end{aligned}$$

Thus by a change of variable

$$2^{jn}\int_{\mathbb{R}^n}|h(2^jy)y^s|dy = 2^{-js}\int_{\mathbb{R}^n}|h(y)y^s|dy \leq 2^{-js}\int_{\mathbb{R}^n}\frac{|y|}{(1+|y|^k)^2}dy < \infty$$

for k large enough. Hence the claim is proved. □

Lemma 2.4.9. *Let s be in $(0,1)$, $r \in \mathbb{R}$ and $(u,v) \in \mathcal{C}^s \times \mathcal{C}^r$. Then there exist constant $C_{s,r}$ such that there is a bounded linear operator*

$$R_j : \mathcal{C}^s \times \mathcal{C}^r \to L^\infty : (u,v) \longmapsto \Delta_j T_u v - u\Delta_j v$$

which satisfies

$$\|R_j\|_{L^\infty} \leq 2^{-j(s+r)}C_{s,r}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^r}.$$

Proof. Since

$$\mathcal{F}(T_u\Delta_i v) = \sum_{j'\geq -1} \mathcal{F}\Big(\mathcal{F}^{-1}(S_{j'}u) * \mathcal{F}^{-1}(\Delta_{j'}\Delta_i v)\Big) \Longrightarrow \operatorname{Supp} \mathcal{F}(T_u\Delta_i v) \subseteq 2^i\mathcal{C}$$

for an annulus $\mathcal{C}$. By partition of unity, if $|i-j|\leq 1$, then we have

$$\Delta_j T_u \Delta_i v = 0 \quad \text{and} \quad \Delta_j\Delta_i v = 0 \quad j\geq -1.$$

Thus

$$\begin{aligned}\Delta_j T_u v &= \sum_{|i-j|\leq 1} \Delta_j(T_u\Delta_{j'}v)\\ &= \sum_{|i-j|\leq 1} \Delta_j(u\Delta_i v) - \sum_{|i-j|\leq 1} \Delta_j T_{\Delta_i v} u - \sum_{|i-j|\leq 1} \Delta_j R(u,\Delta_i v)\\ &= \sum_{|i-j|\leq 1} [\Delta_j,u]\Delta_i v + \sum_{|i-j|\leq 1} u\Delta_j\Delta_i v - \sum_{|i-j|\leq 1} \Delta_j T_{\Delta_i v}u - \sum_{|i-j|\leq 1} \Delta_j R(u,\Delta_i v).\end{aligned}$$

And we conclude from Lemma 2.4.8 and Proposition 2.4.1 that

$$\begin{aligned}\|R_j\|_{L^\infty} &= \Big\|\Delta_j T_u v - \sum_{|i-j|\leq 1} u\Delta_j\Delta_i v\Big\|_{L^\infty}\\ &\leq \sum_{|i-j|\leq 1} \|[\Delta_j,u]\Delta_i v\|_{L^\infty} + \sum_{|i-j|\leq 1} \|\Delta_j T_{\Delta_i v}u\|_{L^\infty} + \sum_{|i-j|\leq 1} \|\Delta_j R(u,\Delta_i v)\|_{L^\infty}\\ &\leq 2^{-js}C_s\|u\|_{\mathcal{C}^s} \sum_{|i-j|\leq 1} \|\Delta_i v\|_{L^\infty} \leq 2^{-j(s+r)}C_{s,r}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^r}.\end{aligned}$$

This concludes the proof. □

We can now prove our main commutator lemma.

Lemma 2.4.10. *Suppose that u,v,w are smooth functions on $\mathbb{R}^n$ and let s,r,σ be real numbers such that $s+\sigma+\rho>0$ and $\sigma+\rho<0$. Then there exists trilinear operator*

$$\Gamma : \mathcal{C}^s\times\mathcal{C}^\sigma\times\mathcal{C}^\rho \to \mathcal{C}^{s+\sigma+\rho} : (u,v,w) \longmapsto R(T_u v,w) - uR(v,w)$$

such that

$$\|\Gamma(u,v,w)\|_{\mathcal{C}^{r+\sigma}} \leq C_{s,\sigma,\rho}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^r}\|w\|_{\mathcal{C}^\sigma}.$$

Proof. Suppose that u,v and w are Schwartz functions. Consider the required operator in the decomposition

$$R(T_u v,w) - uR(v,w) = \sum_{j,j'\geq -1}\sum_{|\nu-j|\leq 1} \Delta_\nu T_{\Delta_{j'}u} v\Delta_j w - u\Delta_\nu v\Delta_j w.$$

Since the Fourier transform of $\Delta_\nu T_{\Delta_{j'}u}v$ is supported outside of a ball $2^\nu B$, there exists positive integer N such that $\Delta_\nu T_{\Delta_{j'}v}u = 1_{\nu\geq j'+N}\Delta_\nu T_{\Delta_{j'}v}u$. Thus by using Lemma 2.4.9 we deduce that

$$\begin{aligned}\Gamma(u,v,w) &= \sum_{j,j'\geq -1}\sum_{|\nu-j|\leq 1} 1_{\nu\geq j'+N}\Big(\Delta_{j'}u\Delta_\nu v + R_\nu(\Delta_{j'}u,v)\Big)\Delta_j w - \Delta_{j'}u\Delta_\nu v\Delta_j w\\ &= \sum_{j,j'\geq -1}\sum_{|\nu-j|\leq 1} 1_{\nu\geq j'+N}R_\nu(\Delta_{j'}u,v)\Delta_j w - 1_{\nu<j'+N}\Delta_{j'}u\Delta_\nu v\Delta_j w.\end{aligned}$$

For the second sum, we observe that

$$\Big\| \sum_{j\geq -1} \sum_{|\nu-j|\leq 1} 1_{\nu<j'+N} \Delta_{j'}(u\Delta_\nu v)\Delta_j w \Big\|_{L^\infty} \leq C_N 2^{-j's}\|u\|_{\mathcal{C}^s} \sum_{\nu=-1}^{j'+N-1} 2^{-\nu(\sigma+\rho)}\|v\|_{\mathcal{C}^\sigma}\|w\|_{\mathcal{C}^\rho}$$
$$\leq C_N 2^{(N-1)(\sigma+\rho)} 2^{-j'(s+\sigma+\rho)}\|u\|\|v\|_{\mathcal{C}^\sigma}\|w\|_{\mathcal{C}^\rho}.$$

On the other hand, for the first sum we have

$$\Big\| \sum_{j'\geq -1} \sum_{|\nu-j|\leq 1} 1_{\nu\geq j'+N} R_\nu(\Delta_{j'}u, v)\Delta_j w \Big\|_{L^\infty} \leq \Big\| \sum_{|\nu-j|\leq 1} R_\nu\Big(\sum_{j'=-1}^{\nu-N} \Delta_{j'}u, v\Big)\Delta_j w \Big\|_{L^\infty}$$
$$\leq \sum_{|\nu-j|\leq 1} 2^{-\nu(s+\sigma)} \Big\| \sum_{j'=1}^{\nu-N} \Delta_{j'}u \Big\|_{\mathcal{C}^s} \|v\|_{\mathcal{C}^\sigma} 2^{-j\rho}\|w\|_{\mathcal{C}^\rho} \leq C_{s,\sigma,N} 2^{-j(s+\sigma+\rho)}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^\sigma}\|w\|_{\mathcal{C}^\rho}.$$

Since $1_{\nu\geq j'+N}R_(\nu)(\Delta_{j'}u, v)\Delta_j w$ is supported in a ball $2^j\tilde{B}$, the required estimate follows from Lemma 2.3.2.

Choose now $(s', \sigma', \rho') \in (0,1)\times\mathbb{R}^2$ such that $s' < s, \sigma' < \sigma$ and $\rho' < \rho$ and consider the continuous extension of Γ into $\mathcal{C}^{s'}\times\mathcal{C}^{\sigma'}\times\mathcal{C}^{\rho'}$. Because $\mathcal{C}^s\times\mathcal{C}^\sigma\times\mathcal{C}^\rho$ is contained in the closure of the smooth functions in $\mathcal{C}^{s'}\times\mathcal{C}^{\sigma'}\times\mathcal{C}^{\rho'}$, we obtain the required extension, and the corresponding bound follows from

$$\|\Gamma(u,v,w)\|_{\mathcal{C}^{s+\sigma+\rho}} = \limsup_{(s',\sigma',\rho')\to(s,\sigma,\rho)} \|\Gamma(u,v,w)\|_{\mathcal{C}^{s'+\sigma'+\rho'}}$$
$$\leq C_{s,\sigma,\rho} \limsup_{(s',\sigma',\rho')\to(s,\sigma,\rho)} \|u\|_{\mathcal{C}^{s'}}\|v\|_{\mathcal{C}^{\sigma'}}\|w\|_{\mathcal{C}^{\rho'}} = \|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^\sigma}\|w\|_{\mathcal{C}^\rho}$$

as desired. □

With the above commutator at hand and the paralinearisation estimate, we can also look at action of continuous functions on the remainder operator.

Lemma 2.4.11. *Let s be in $(0,1)$ and $\sigma \in (0,s]$. Suppose $\rho < 0$ is chosen such that $s+\sigma+\rho > 0$ and $s+\sigma < 0$. Then for every $f \in C_b^{1,\sigma/s}$ there exists a locally bounded operator*

$$\Pi_f : \mathcal{C}^s\times\mathcal{C}^\sigma \to \mathcal{C}^{s+\sigma+\rho} : (u,v) \longmapsto R(f\circ u, v) - f'\circ u R(u,v)$$

such that

$$\|\Pi_f(u,v)\|_{\mathcal{C}^{s+\sigma+\rho}} \leq C_{s,\sigma,\rho}\|f\|_{C_b^{1,\sigma/s}}(1+\|u\|_{\mathcal{C}^s}^{1+\sigma/s})\|v\|_{\mathcal{C}^\rho}.$$

If instead f is in $C_b^{2,\sigma/s}$, then Π_f is locally Lipschitz continuous in the sense that

$$\|\Pi_f(u_1,v_1) - \Pi_f(u_2,v_2)\|_{\mathcal{C}^{s+\sigma+\rho}}$$
$$\leq C_{s,\sigma,\rho}\|F\|_{C_b^{2,\sigma/s}}(1+(\|u_1\|_{\mathcal{C}^s}+\|v_1\|_{\mathcal{C}^s})^{1+\sigma/s}+\|v_2\|_{\mathcal{C}^\rho})(\|u_1-u_2\|_{\mathcal{C}^s}+\|v_1-v_2\|_{\mathcal{C}^\rho}).$$

Proof. By applying the commutator estimate 2.4.10 and a paralinearisation 2.4.4, we have

$$\Pi_f(u,v) = R(f\circ u, v) - f'\circ u R(u,v)$$
$$= R(T_{f'\circ u}u, v) + R(R_f u, v) - f'\circ u R(u,v) = \Gamma(f'\circ u, u, v) + R(R_f u, v).$$

Thus the estimate follows from Lemma 2.4.10 and Proposition 2.4.4. The Lipschitz continuous case follows analogously. □

Lemma 2.4.12. *Let s be in $(0,1)$ and $\nu < 0$ be such that $2s+\nu > 0$ and $s+\nu < 0$. Then there exists a bounded trilinear operator*

$$\Sigma : \mathcal{C}^s \times \mathcal{C}^s \times \mathcal{C}^\nu \to \mathcal{C}^{2s+\nu} : (u,v,w) \to R(uv,w) - uR(v,w) - vR(u,w).$$

Proof. It suffices to note that

$$\Sigma(u,v,w) = \Gamma(u,v,w) + \Gamma(v,u,w) + R(R(u,v),w)$$

and then apply Lemma 2.4.10 to conclude the proof. □

2.5 Space-Time Besov Spaces

When solving partial differential equations it is necessary to consider distributions in space and time. For $(\sigma, s) \in \mathbb{N} \times \mathbb{R}$ and $T > 0$ recall the classical spaces

$$C_T^\sigma(L^\infty) \stackrel{\text{def}}{=} C([0,T];L^\infty) \quad \text{and} \quad \|u\|_{C_T^\sigma(L^\infty)} \stackrel{\text{def}}{=} \sum_{k\le\sigma} \sup_{t\le T} \|\partial_t^k u(t)\|_{L^\infty}$$
$$C_T^\sigma(\mathcal{C}^s) \stackrel{\text{def}}{=} C([0,T];\mathcal{C}^s) \quad \text{and} \quad \|u\|_{C_T^\sigma(\mathcal{C}^s)} \stackrel{\text{def}}{=} \sum_{k\le\sigma} \sup_{t\le T} \|\partial_t^k u(t)\|_{\mathcal{C}^s}$$

as well as

$$C_T^{[s],s-[s]}(L^\infty) \stackrel{\text{def}}{=} C^{[s],s-[s]}([0,T];L^\infty) \quad \text{with}$$
$$\|u\|_{C_T^{[s],s-[s]}(L^\infty)} \stackrel{\text{def}}{=} \|u\|_{C_T^{[s]}(L^\infty)} + \sup_{\substack{t\neq t' \\ t,t'\le T}} \frac{\|\partial^{[s]}u(t) - \partial^{[s]}u(t')\|_{L^\infty}}{|t-t'|^{s-[s]}}.$$

We say that $u \in C^\sigma(\mathcal{C}^s)$ if u is in $C_T^\sigma(\mathcal{C}^s)$ for every $T > 0$. The *heat operator* is

$$\mathcal{L} \stackrel{\text{def}}{=} \partial_t - \Delta.$$

We will often look for solutions to differential equations in the normed space $(\mathcal{L}_T^s, \|\cdot\|_{\mathcal{L}_T^s})$:

$$\mathcal{L}_T^s \stackrel{\text{def}}{=} C_T^{[s/2],s/2-[s/2]}(L^\infty) \cap C_T(\mathcal{C}^s) \quad \text{and} \quad \|u\|_{\mathcal{L}_T^s} \stackrel{\text{def}}{=} \|u\|_{C_T^{[s/2],s/2-[s/2]}(L^\infty)} + \|u\|_{C_T(\mathcal{C}^s)}.$$

Likewise, we say that $u \in \mathcal{L}^s$ if u is in $\mathcal{L}_T^s$ for every $T > 0$.

2.5.1 Time Regularisation of Paraproducts

For our purpose, it would also be convenient to introduce the following modifications. Let $\tilde{\varphi} \in C^\infty(\mathbb{R};\mathbb{R}^+)$ be nonnegative with compact support in $\mathbb{R}^+$ and total mass 1. For all $s \in \mathbb{R}$ and $j \ge -1$ define

$$\tilde{S}_j : \begin{cases} C_T(\mathcal{C}^s) \longrightarrow C_T(\mathcal{C}^s) \\ \quad u \quad \longmapsto \displaystyle\sum_{j'\le j-2} \int_{\mathbb{R}} 2^{2(j+1)} \tilde{\varphi}\big(2^{2(j+1)}(t-t')\big) \Delta_{j'} u\Big(\max\Big\{\min\{t',T\},0\Big\}\Big) dt' \end{cases}$$

and the corresponding paraproduct

$$\tilde{T}_u v \stackrel{\text{def}}{=} \sum_{j \geq -1} \tilde{S}_{j-1} u \Delta_j v.$$

Continuity estimates for Paraproducts in space-time Besov spaces hold analogously.

Proposition 2.5.1. *For any $(s, r) \in \mathbb{R}^- \times \mathbb{R}$ and $T > 0$, for all $(u, v) \in C_T(\mathcal{C}^s)^2$ we have*

1. $\|\tilde{T}_u v(T)\|_{\mathcal{C}^r} \leq C\|u\|_{C_T(L^\infty)}\|v(T)\|_{\mathcal{C}^r}$,
2. $\|\tilde{T}_u v(T)\|_{\mathcal{C}^{s+r}} \leq C'\|u\|_{C_T(\mathcal{C}^s)}\|v(T)\|_{\mathcal{C}^r}$.

Proof. With the modification, for each T the Fourier transform of $(\tilde{S}_{j-1} u \Delta_j v)(T)$ is still supported in $2^j \tilde{\mathcal{C}}$. A direct calculation yields

$$\|(\tilde{S}_{j-1} u)(T)\|_{L^\infty} \leq C 2^{2(j+1)} \|S_{j-1} u\|_{C_T(L^\infty)} \int_{\mathbb{R}} |\tilde{\varphi}(2^{2(j+1)}(t'))| dt' \leq C\|u\|_{C_T(L^\infty)}.$$

Therefore, the identity

$$\|(\tilde{S}_{j-1} u \Delta_j v)(T)\|_{L^\infty} \leq 2^{-jr} C\|u\|_{C_T(L^\infty)}\|u(T)\|_{\mathcal{C}^r}$$

proves the first claim. The second claim follows similarly from another modified argument of Proposition 2.4.1. □

The following estimates will describes the commutation between various paraproducts and the heat operator.

Lemma 2.5.2. *Suppose that $(r, T) \in \times\mathbb{R} \times (0, \infty)$. If (u, v) is in ${\mathcal{L}_T^s}^2$, then we have*

$$\|(\tilde{T}_u v - T_u v)(T)\|_{\mathcal{C}^{s+r}} \leq C\|u\|_{\mathcal{L}_T^s}\|v(T)\|_{\mathcal{C}^r} \quad \textit{if} \quad s \in (0, 2)$$

$$\|(\mathcal{L}\tilde{T}_u v - \tilde{T}_u \mathcal{L} v)(T)\|_{\mathcal{C}^{s+r-2}} \leq C'\|u\|_{\mathcal{L}_T^s}\|v(T)\|_{\mathcal{C}^r} \quad \textit{if} \quad s \in (0, 1).$$

Proof. Let us frist note that

$$\begin{aligned}
\|(\tilde{S}_j u - S_j u)(T)\|_{L^\infty} &\leq \|S_j u\|_{C_T^{0,s/2}(L^\infty)} \int_{\mathbb{R}} 2^{2(j+1)} \tilde{\varphi}\Big(2^{2(j+1)}(T - t')\Big)|t' - T|^{s/2} dt' \\
&\leq C 2^{-(j+1)(s)} \|u\|_{C_T^{0,s/2}(L^\infty)} \int_{\mathbb{R}} \tilde{\varphi}(t')|t'|^{s/2} dt', \\
\|\partial_t \tilde{S}_j u(T)\|_{L^\infty} &\leq \|S_j u\|_{C_T^{0,s/2}(L^\infty)} \int_{\mathbb{R}} 2^{4(j+1)} \tilde{\varphi}'\Big(2^{2(j+1)}(T - t')\Big)|t' - T|^{s/2} dt' \\
&\leq C 2^{(2-s)(j+1)} \|u\|_{C_T^{0,s/2}(L^\infty)} \int_{\mathbb{R}} \tilde{\varphi}'(t')|t'|^{s/2} dt'.
\end{aligned}$$

Thus, for $s \in (0, 1)$ we readily estimate

$$\|(\tilde{S}_{j-1} u - S_j u)\Delta_j v(T)\|_{L^\infty} \leq C 2^{-j(s+r)} \|u\|_{C_T^{0,s/2}(L^\infty)} \|v(T)\|_{\mathcal{C}^r}.$$

Since

$$\tilde{T}_u v - T_u v = \sum_{j \geq -1} (\tilde{S}_{j-1} u - S_{j-1} u)\Delta_j v$$

by Lemma 2.3.2 this concludes the proof of the first claim. For the second claim, we use that

$$\mathcal{L}\tilde{T}_u v - \tilde{T}_u \mathcal{L} v = \tilde{T}_{\mathcal{L}u} v - \sum_{k\leq n} \tilde{T}_{\partial_k u}\partial_k v$$

Therefore, by what was proved we have

$$\begin{aligned}\|\mathcal{L}\tilde{S}_{j-1}u(T)\|_{L^\infty} &\leq \|\partial_t \tilde{S}_{j-1}u(T)\|_{L^\infty} + \|\tilde{S}_j \Delta u(T)\|_{L^\infty} \\ &\leq C2^{-(s-2)j}(\|u\|_{C_T^{0,s/2}(L^\infty)} + \|u\|_{C_T(C^s)}), \\ \|\tilde{T}_{\partial_k u}\partial_k v\|_{\mathcal{C}^{s+r-2}} &\leq C\|u\|_{C_T(\mathcal{C}^s)}\|v\|_{\mathcal{C}^r}.\end{aligned}$$

Applying Lemma 2.3.2 again concludes the proof. □

2.5.2 A Schauder Estimate

In this subsection we will prove a simple *Schauder estimate.* Recall that the *heat equation* (H) is

$$\begin{cases}\mathcal{L}u = f \\ u_{|t=0} = u_0.\end{cases}$$

defined for $t \geq 0$, forcing function $f(t,x)$ and initial data $u(x)$.

If we have $(u_0, f) \in \mathcal{S} \times C(\mathbb{R}^+; \mathcal{S})$, then it is easy to see by taking Fourier transform in space that (H) is equivalent to the following

$$\begin{cases}\partial_t \hat{u} + |\xi|^2 \hat{u} = \hat{f} \\ \hat{u}_{|t=0} = \hat{u}_0,\end{cases}$$

which is explicitly solvable via the formula

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{u}_0(\xi) + \int_0^t e^{|\xi|^2(t-t')}\hat{f}(t',\xi)dt.$$

It is conventienal to apply the notation

$$u(t,x) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-t')\Delta} f(t',x)dt'.$$

If (H) has no external force, then $e^{t\Delta} : \mathcal{S}' \to \mathcal{S}'$ defines an operator that maps every initial data of (H) to its corresponding solution. To recover analogies to more general (H), we then set for all $f \in \mathcal{S}'$

$$V_t f \stackrel{\text{def}}{=} \int_0^t e^{(t-t')\Delta} f(t',x)dt'.$$

Thus formally the differential operator $\mathcal{L}$ is invertible. The collection $(e^{t\Delta})_{t\geq 0}$ forms the *heat semigroup* and induces the following heat flow estimats.

Lemma 2.5.3. *Let (s,σ) be in $\mathbb{R} \times [0,\infty)$ and $T > 0$. Then for all $u \in \mathcal{C}^s$ we have*

$$\|e^{t\Delta}u\|_{\mathcal{C}^{s+\sigma}} \leq Ct^{-\sigma/2}\|u\|_{\mathcal{C}^s}$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ be a function with value 1 near $\mathcal{C}$. Then

$$e^{t\Delta}\Delta_j u = \mathcal{F}^{-1}e^{-t|\cdot|^2}\phi(2^{-j}\cdot) * \Delta_j u.$$

Thus by Young's inequality,

$$\|e^{t\Delta}\Delta_j u\|_{L^\infty} \leq 2^{-js}\|\mathcal{F}^{-1}e^{-t|2^j\cdot|^2}\phi\|_{L^1}\|u\|_{\mathcal{C}^s}.$$

It follows from a direct calculation that

$$\begin{aligned}
\|\mathcal{F}^{-1}e^{-t|2^j\cdot|^2}\phi\|_{L^1} &\leq \|(1+|\cdot|^2)^n\mathcal{F}^{-1}e^{-t|2^j\cdot|^2}\phi\|_{L^\infty}\|(1+|\cdot|^2)^{-n}\|_{L^1}\\
&= \|\mathcal{F}^{-1}(1+\Delta)^n e^{-t|2^j\cdot|^2}\phi\|_{L^\infty}\|(1+|\cdot|^2)^{-n}\|_{L^1}\\
&\leq \|(1+\Delta)^n e^{-t|2^j\cdot|^2}\phi\|_{L^1}\|(1+|\cdot|^2)^{-n}\|_{L^1}\\
&\leq C(1+2^j t)^{2n}\sup_{|\alpha|\leq\sigma+2n}\|\partial^\alpha e^{-t|2^j\cdot|^2}\|_{L^\infty(\text{supp }\phi)}\|(1+|\cdot|^2)^{-n}\|_{L^1}.
\end{aligned}$$

Since ϕ is supported in $\mathcal{C}$, there exists a minial j_0 such that $2^{j_0}t|x| \geq 1$ for all $x \in \text{Supp }\phi$. Then, by using the decay for the expoential function, there exists positive constant C such that

$$\sup_{|\alpha|\leq\sigma+2n}|\partial^\alpha e^{-|x|^4}| \leq C(1+|x|^2)^{-\sigma-2n} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

So for all $j \geq j_0$

$$\|\mathcal{F}^{-1}e^{-t|2^j\cdot|^2}\phi\|_{L^1} \leq C(1+2^j t)^{2n}(1+2^j t)^{-\sigma-2n} \leq C2^{-j\sigma}t^{-\sigma}.$$

On the other hand, for $0 \leq j < j_0$ we have the obvious estimates

$$\begin{aligned}
\|e^{t\Delta}\Delta_j u\|_{L^\infty} &\leq \|\mathcal{F}^{-1}e^{-t|\cdot|^2}\|_{L^1}\|\Delta_j u\|_{L^\infty}\\
&\leq 2^{-js}\|\mathcal{F}^{-1}e^{-t|\cdot|^2}\|_{L^1}\|u\|_{\mathcal{C}^s}\\
&= (t2^j)^\sigma t^{-\sigma}2^{-j(s+\sigma)}\|\mathcal{F}^{-1}e^{-t|\cdot|^2}\|_{L^1}\|u\|_{\mathcal{C}^s}\\
&\leq \||\cdot|^{-1}\|_{L^\infty(\mathcal{C})}t^{-\sigma}2^{-j(s+\sigma)}\|\mathcal{F}^{-1}e^{-t|\cdot|^2}\|_{L^1}\|u\|_{\mathcal{C}^s}.
\end{aligned}$$

Fianlly the case $j = -1$ is trivial. This concludes the proof. □

We have the following technical lemma.

Lemma 2.5.4. *Let s be in $(0,1)$ and $t \geq 0$. Then for all $u \in \mathcal{C}^s$ we have*

$$\|(e^{t\Delta} - Id)u\|_{L^\infty} \leq Ct^{s/2}\|u\|_{\mathcal{C}^s}.$$

Proof. It suffices to compute $e^{t\Delta}$ in terms of convolution and with the identification from Proposition 2.3.6 to write

$$\begin{aligned}
|(e^{t\Delta} - \text{Id})u| &= t^{-n/2}\Big|\int_{\mathbb{R}^n}(\mathcal{F}^{-1}e^{-|\cdot|^2})\big(t^{-1/2}(x-y)\big)\big(u(y)-u(x)\big)dy\Big|\\
&\leq 2^{s/2}\|u\|_{\mathcal{C}^s}\int_{\mathbb{R}^n}|\mathcal{F}^{-1}e^{-|\cdot|^2}(y)||y|^s dy
\end{aligned}$$

as desired. □

Let us now prove the second part of the estimate.

Proposition 2.5.5. *For all* $(s,T) \in (0,2) \times \mathbb{R}^+$ *and* $u \in C_T(\mathcal{C}^{s-2})$, *we have*

$$\|V_t u\|_{\mathcal{L}_T^s} \leq C\|u\|_{C_T(\mathcal{C}^{s-2})}$$

Proof. Consider for $j \geq 0$, $T \in (0,\infty)$ and $\rho \in [0, T/2) \cap [0,1]$ the decomposition

$$\Delta_j V_T u = \int_0^{T/2-\rho} \Delta_j e^{(T-t)\Delta} u(t,x)dt + \int_{T/2-\rho}^{T} \Delta_j e^{(T-t)\Delta} u(t,x)dt.$$

By Lemma 2.5.3 and a change of variable, estimation for the second term follows from

$$\begin{aligned}
\Big\| \int_{T/2-\rho}^{T} \Delta_j e^{(T-t)\Delta} u(t,x)dt \Big\|_{L^\infty} &\leq \lim_{\epsilon\to 0} 2^{-j(s+2\epsilon)} \int_{T/2-\rho}^{T} \|e^{(T-t)\Delta}u(t)\|_{\mathcal{C}^{s+2\epsilon}} dt \\
&\leq \lim_{\epsilon\to 0} C 2^{-j(s+2\epsilon)} \|u\|_{C_T(\mathcal{C}^{s-2})} \int_{T/2-\rho}^{T} t^{-1-\epsilon} dt \\
&= \lim_{\epsilon\to 0} C 2^{-j(s+2\epsilon)} \|u\|_{C_T(\mathcal{C}^s)} T^{-\epsilon} \int_{1/2-\rho/T}^{1} t^{-1-\epsilon} dt \\
&\leq \lim_{\epsilon\to 0} C 2^{-js} \Big(\frac{T}{2}+\rho\Big)\Big(\frac{T}{2}-\rho\Big)^{-1} \Big(2^{-2j}(T-\rho)\Big)^{\epsilon} \|u\|_{C_T(\mathcal{C}^s)}.
\end{aligned}$$

As for the first term, we can bound

$$\begin{aligned}
\Big\| \int_0^{T/2-\rho} \Delta_j e^{(T-t)\Delta} u(t,x)dt \Big\|_{L^\infty} &\leq 2^{-j(s-2)} \|u\|_{C_T(\mathcal{C}^{s-2})} (T/2-\rho) \\
&\leq 2^{-js} \|u\|_{C_T(\mathcal{C}^{s-2})}
\end{aligned}$$

where in the last step we adjust ρ sufficiently large such that $T/2-\rho = 2^{-js}$ for $2^{-js} \leq T/2-\rho$ or by bounding $T/2-\rho \leq 2^{-js}$ otherwise. The case $j=-1$ is trivial. This proves

$$\|V_t u\|_{C_T(\mathcal{C}^s)} \leq C\|u\|_{C_T(\mathcal{C}^{s-2})}. \tag{2.8}$$

Now notice that for all $0 \leq t \leq t' \leq T$, we have

$$V_t u - V_{t'} u = (e^{(t-t')\Delta} - \mathrm{Id}) V_t u + \int_t^{t'} e^{(t'-t'')\Delta} u(t'') dt''.$$

Therefore by an application of Lemma 2.5.4 and identity (2.8), we get

$$\begin{aligned}
\|V_t u - V_{t'} u\|_{L^\infty} &\leq \|(e^{(t-t')\Delta} - \mathrm{Id}) V_t u\|_{L^\infty} + \int_t^{t'} \|e^{(t'-t'')\Delta} u(t'')\|_{L^\infty} dt'' \\
&\leq C\Big(|t-t'|^{s/2} \|u\|_{C_T(\mathcal{C}^{s-2})} + \int_t^{t'} |t'-t''|^{(s-2)/2} \|u(t'')\|_{s-2} dt''\Big) \\
&\leq C|t-t'|^{s/2} \|u\|_{C_T(\mathcal{C}^{s-2})}.
\end{aligned}$$

which concludes the proof. □

We can now formally state the our Schauder estimate

Proposition 2.5.6. *Let s be in $(0,2)$ and $f \in C(\mathcal{C}^{s-2})$. Suppose u satisfies (H) with external force f, such that u has initial condition $u_0 \in \mathcal{C}^s$. Then there exists a unique solution $u \in \mathcal{L}^s$ such that*

$$u = e^{t\Delta}u_0 + V_t f.$$

Moreover, for every $T \in \mathbb{R}^+$, u satisfies the estimate

$$\|u\|_{\mathcal{L}^s_T} \leq C_s(\|u_0\|_{\mathcal{C}^s} + \|f\|_{C_T(\mathcal{C}^{s-2})}).$$

Combining the commutator lemma with the Schauder estimate, we are able to obtain our first corollary, which control the regularised paraproduct in the $\mathcal{L}^s_T$ spaces rather than just $C_T(\mathcal{C}^s)$ spaces.

Corollary 2.5.7. *Let (s,r) be in $(0,2)\times\mathbb{R}_+$, $T > 0$ and $(u, v, \mathcal{L}v) \in \mathcal{L}^r_T \times C_T(\mathcal{C}^s) \times C_T(\mathcal{C}^{s-2})$. Then there exists positive constant C such that*

$$\|\tilde{T}_u v\|_{\mathcal{L}^s_T} \leq C\|u\|_{\mathcal{L}^r_T}(\|v\|_{C_T(\mathcal{C}^s)} + \|\mathcal{L}v\|_{C_T(\mathcal{C}^{s-2})}).$$

Proof. It suffies to apply Proposition 2.5.5 and Lemma 2.5.2 along with the Proposition 2.5.1 directly to see that

$$\begin{aligned}
\|\tilde{T}_u v\|_{\mathcal{L}^s_T} &\leq C(\|\tilde{T}_u v(0)\|_{\mathcal{C}^s} + \|\mathcal{L}\tilde{T}_u v\|_{C_T\mathcal{C}^{s-2}}) \\
&\leq C(\|\mathcal{L}\tilde{T}_u v - \tilde{T}_u\mathcal{L}v\|_{C_T(\mathcal{C}^{s+r-2})} + \|\tilde{T}_u\mathcal{L}v\|_{C_T(\mathcal{C}^{s-2})}) \\
&\leq C\|u\|_{\mathcal{L}^r_T}(\|v\|_{C_T(\mathcal{C}^s)} + \|\mathcal{L}v\|_{C_T(\mathcal{C}^{s-2})})
\end{aligned}$$

as required. □

Finally, we note that for the spaces $\mathcal{L}^s$, $s \in (0,2)$, it is possible to pass onto a larger space by gaining a small scaling factor.

Lemma 2.5.8. *Let s be in $(0,2)$, $T > 0$ and $u \in \mathcal{L}^s_T$. Then for all $r \in (0,s]$ there exists $C > 0$ such that*

$$\|u\|_{\mathcal{L}^r_T} \leq C\|u(0)\|_{\mathcal{C}^r} + T^{(s-r)/2}\|u\|_{\mathcal{L}^s_T}.$$

The proof of this Lemma is inherently simple but requires the application of an interpolation result, for which there is no room to present. Hence the proof is omitted.

2.6 Extensions and Special Cases

We briefly discuss some necessary extensions to the definition of the space $\mathcal{C}^s$.

2.6.1 Multidimensional Hölder-Besov Spaces

In Definition 2.3.1 we only looked at the space $\mathcal{C}^s(\mathbb{R}^n;\mathbb{C})$ concerning one-dimensional tempered distributions, but we could also extend the concept of tempered distributions to higher dimensions. Indeed, one could define analogously the space $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^d)$ with topology generated by the semi-norms $\|\cdot\|_{k,\mathcal{S}(\mathbb{R}^n;\mathbb{C}^d)}$ exactly as in the one-dimensional case. The dual space $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^d)$ then consists of functionals

$$\langle u,\phi\rangle = \sum_{1\leq j\leq d}\langle u_j,\phi_j\rangle,\quad \phi=(\phi_1,...,\phi_d)\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^d),$$

where each ϕ_j is a one-dimensional tempered distribution with action

$$\langle u_j,\phi\rangle \stackrel{\text{def}}{=} \langle u,\phi_j e_j\rangle.$$

The Fourier transform of u acts on every such components. Moreover, each u corresponds to a vector representation $u=(u_1,...,u_d)$ where the elements are one-dimensional tempered distributions.

For every $u\in\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^d)$, the Paley-Littlewood theory remains true, since for scalar valued functions φ and χ, the tempered distributions $\varphi(2^{-j}\xi)\hat{u}(\xi)$ and $\chi(\xi)\hat{u}(\xi)$ still make sense, and therefore the convergence

$$\lim_{N\to\infty}\sum_{j\leq N}\Delta_j u$$

can be interpreted component wise in $\mathcal{S}'(\mathbb{R}^n;\mathbb{C})$. The multi-dimensional Hölder-Besov spaces can be defined by the same norm

$$\|u\|_{\mathcal{C}^s(\mathbb{R}^n;\mathbb{R}^d)} \stackrel{\text{def}}{=} \left\|(2^{js}\|\Delta_j u\|_{L^\infty(\mathbb{R}^n;\mathbb{C}^d)})_{j\geq -1}\right\|_{\ell^\infty}$$

where $\|\Delta_j u\|_{L^\infty(\mathbb{R}^n;\mathbb{C}^d)} = \sup_{x\in\mathbb{R}^n}|\Delta_j u(x)|$ is defined in terms of the d-dimensional modulus.

Let $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)$ be the space of d by m $\mathbb{R}^n$-matrices and $F:\mathbb{R}^n\to\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)$ be a family of vector fields with component functions $(f_{ij})_{1\leq i\leq d,1\leq j\leq m}$. Every F can be written in the form

$$F=\sum_{i\leq d,j\leq m} f_{ij}\ e_i\otimes e_j.$$

with $e_i\otimes e_j$ being the pure tensor basis of $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)$. With this expression, one easily sees that the Littlewood-Paley theory holds for products induced by application of F on vectors in $\mathbb{R}^d$. Moreover, if $n=1$ and $u:\mathbb{R}\to\mathbb{R}^\ell$, then the paralinearisation as well as the boundedness results proved in subsection 1.4.2. extends to this multi-dimensional case, where the notion F' is replaced by the Fréchet derivative.

2.6.2 The Periodic Case

Consider instead functions $u:\mathbb{T}^n\to\mathbb{R}$ defined on the n-dimensional torus where

$$\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/2\pi\mathbb{Z}.$$

Since $\mathbb{T}^n$ is compact, we can analogously define tempered distributions $\mathcal{S}'(\mathbb{T}^n)$ to be the dual space of $C^\infty(\mathbb{T}^n)$. Moreover, there is a distinct theory of Fourier transform for the space of such functions, where the map is defined by

$$\mathcal{F}(\phi)(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{T}^n}\phi(x)e^{-ix\cdot\xi}dx \quad\text{and}\quad \mathcal{F}^{-1}(\phi)_{\xi\in\mathbb{Z}^n}(x) \stackrel{\text{def}}{=} \sum_{\xi\in\mathbb{Z}^n}\phi(\xi)e^{-ix\cdot\xi}.$$

This defines a bijection between $C^\infty(\mathbb{T}^n)$ and functions on $\mathbb{Z}^n$ which are at most polynomial growth.

It is well-known that every function defined on the torus can be uniquely identified as a periodic function $u : \mathbb{R}^n \to \mathbb{R}$. Under this identification, the Littlewood-Paley decomposition still applies and we can likewise define Hölder-Besov spaces $\mathcal{C}^s(\mathbb{T}^n)$. In particular, everything that was proved for $\mathcal{C}^s(\mathbb{R}^n)$ applies to the case of $\mathcal{C}^s(\mathbb{T}^n)$.

2.7 References and Remarks

Most of the contents on $\mathcal{C}^s(\mathbb{R}^n)$ were borrowed from [1], where the the more general Besov spaces $B^s_{p,r}$ for all $(s,p,r) \in \mathbb{R}^3$, are considered. The Littlewood-Paley theory first appeared in the context of one-dimensional Fourier series. The current presentation is rather restrictive by considering only the cruder Littlewood-Paley decomposition, but proves to be sufficient for tackling most problems related to nonlinear partial differential equations. Paradifferential calculus was first invented by J.-M. Bony in more general context for proving a priori estimates of quasilinear hyperbolic partial differential equations. We considered here the discrete version, which was due to P. Gérard and J. Rauch. The commutator estimates were partially taken from [2] for the explicit purpose of decomposing the non-linearity which will be seem in Chapter 2. The Schauder estimate is classical and was also considered in [2], but we only looked at a simplified case which is suitable for the treatment in [6]. The special spaces $\mathcal{L}$ is studied for the purpose of Chapter 3 where we will use considerably many times the smoothing properties of the heat equation. The higher dimensional spaces $\mathcal{C}^s(\mathbb{R}^n; \mathbb{R}^d)$ will be needed in Chapter 2 and the periodic Hölder-Besov spaces $\mathcal{C}^s(\mathbb{T}^n)$ will be looked at in Chapter 3, where we should in principle consider partial differential equations defined with periodic boundary conditions.

Chapter 3

Ordinary Differential Equations Driven by Rough Signals

Let $u \in C(\mathbb{R};\mathbb{R}^n)$ be continuous vector valued function, $X \in \mathcal{C}^s(\mathbb{R};\mathbb{R}^d)$ be vector valued distribution for $s \in (1/3, 1/2)$. Suppose that $F : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^d;\mathbb{R}^n)$ is a family of vector fields which is at least twice differentiable and bounded. The prototypical example for an application of the paracontrolled distribution is to solve the *rough differential equation* (*RDE*):

$$\begin{cases} \dfrac{d}{dt}u = F(u(t))dX \\ u_{|t=0} = u_0, \quad u_0 \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where we interpret dX to be the derivative of X taken in the distributional sense. Since the time derivative of u loses one order of regularity, the regularity of X conditions that u behaves no better than $\mathcal{C}^s(\mathbb{R};\mathbb{R}^n)$. In particular $F(u) \in \mathcal{C}^s(\mathbb{R};\mathcal{L}(\mathbb{R}^d;\mathbb{R}^n))$. The distribution X is the *signal* and solutions to (3.1) are said to be *driven by* X.

The theory of paracontrolled analysis originated from solving equations of the form (3.1). The classical approach to solve such equations involve systematic understandings of integration. In the settings where the signal is moderately irregular, one could apply the construction by L.C. Young. But in the probabilistic setting, the signal X is usually taken to be a variate of the Brownian motion, which is just smooth enough to be in $\mathcal{C}^{1/2-}$[1] almost surely. Since this level of regularity cannot be treated in such a way, further analysis would be required.

This chapter is devoted to the construction of a subspace of $\mathcal{C}^s$ for $s \in (1/3, 1/2)$, the elements of which are compatible with the necessary structure for the solutions to (3.1), and show the existence of a unique weak solution to (3.1) in such a space provided the signal X can be suitably regularised.

3.1 Integration along Irregular Paths

Our goal in this section is to construct a reasonable notion of definite integral driven by elements in $\mathcal{C}^{s-1}(\mathbb{R};\mathbb{R})$[2] with $s \in (1/3, 1)$. To motivate our theory, we recover Young's theorem in Hölder-Besov spaces with regularity index $s \in [1/2, 1)$ and an one dimensional time index.

[1] This denotes all spaces $\mathcal{C}^s$ for which we have $s < 1/2$.

[2] To avoid abuse of notation we will consider one-dimensional distributions. Integrals of system can be naturally extended.

We begin with a slightly more general construction, which is actually more than what we would need.

Definition 3.1.1. Let s,r be in $\mathbb{R}$ and $(u,v)\in\mathcal{C}^s(\mathbb{R};\mathbb{R})\times\mathcal{C}^r(\mathbb{R};\mathbb{R})$. The *integral of u driven by v* is

$$\int_0^t udv \stackrel{\text{def}}{=} \sum_{j\geq -1}\int_0^t S_{j-1}ud\Delta_j v + \sum_{j\geq -1}\int_0^t \Delta_j udS_{j-1}v + \sum_{|j-k|\leq 1}\int_0^t \Delta_j ud\Delta_k v. \tag{3.2}$$

Let first state the theorem of Young.

Theorem 3.1.2. *Let (s,r) be in $[1/2,1)^2$ and $(u,v)\in\mathcal{C}^s([-T,T];\mathbb{R})\times\mathcal{C}^r([-T,T];\mathbb{R})$. Then the function defined by (3.2) satisfies*

$$\left\|\int_0^t udv\right\|_{\mathcal{C}^r} \leq (1+T)C_{s,r}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^r}.$$

Moreover, we have

$$\left\|\int_0^t udv - T_u v\right\|_{\mathcal{C}^{s+r}} \leq C_{s,r}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^r}.$$

The first part of Young's theorem is a consequence of the following *lifting estimate.*

Proposition 3.1.3. *Let (s,r) be in $(0,1)^2$ such that $s+r\geq 1$ and $(u,v)\in\mathcal{C}^s(\mathbb{R};\mathbb{R})\times\mathcal{C}^r(\mathbb{R};\mathbb{R})$. Suppose that udv has compact support, then the element defined by (3.2) is a function in $\mathcal{C}^r$ which satisfies*

$$\frac{d}{dt}\int_0^t udv = udv \quad \textit{and} \quad \int_0^0 udv = 0, \tag{3.3}$$

where the derivative is taken in the distributional sense. Moreover, for any $T>0$ and $\phi\in\mathcal{D}([-T,T])$, there exists constant $C_{s,r}$ such that

$$\left\|\int_0^t udv\right\|_{\mathcal{C}^r} \leq (1+T)C_{s,r}\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^r}$$

Proof. Let $(i\xi)^{-1}(D)$ be the Fourier multiplier with symbol $(i\xi)^{-1}$. Recall that we have

$$R(u,dv) = \sum_{j\geq -1} R_j(u,dv) \quad \text{where} \quad R_j(u,dv) = \sum_{|\nu|\leq 1}\Delta_{j-\nu}ud\Delta_j v.$$

If $j\geq 0$, then since the Fourier transform of $\Delta_{j-\nu}ud\Delta_j v$ is supported in $2^{j-\nu}\mathcal{C}+2^j\mathcal{C}$, we can proceed as before with an application of the Bernstein Lemma 2.2.4 which yields

$$\left|\sum_{|\nu|\leq 1}\int_{t'}^t \Delta_{j-\nu}ud\Delta_j v\right| \leq \sum_{|\nu|\leq 1} 2^{-j}(1-2^{-\nu})^{-1}\|u\|_{L^\infty}\|\Delta_j dv\|_{L^\infty} \leq 2^{-jr}C\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}.$$

Otherwise, we can apply a direct estimation to see that

$$\left|\sum_{|\nu|\leq 1}\int_{t'}^t \Delta_{j-\nu}ud\Delta_j v\right| \leq C|t-t'|2^{j(1-r)}\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}.$$

Assume $|t-t'| \leq 1$ and choose j_0 such that $2^{-j_0} \leq |t-t'| < 2^{-j_0-1}$. Then clearly

$$\Big|\sum_{j\geq -1}\int_{t'}^{t} R_j(u,dv)dt'\Big| \leq \sum_{j\leq j_0} C_r|t-t'|2^{j(1-r)}\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}} + \sum_{j>j_0} 2^{-jr}C\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}$$
$$\leq C(|t-t'|2^{j_0(1-r)}+2^{-j_0 r})\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}} \leq \frac{C}{2}|t-t'|^r\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}.$$

Since the Fourier transforms of $S_{j-1}ud\Delta_j v$ and $\Delta_j udS_{j-1}v$ are contained in $2^j\tilde{\mathcal{C}}$. Along the same lines we could also get, as in Proposition 2.4.1 that

$$\max\Big\{\Big|\int_{t'}^{t} S_{j-1}d\Delta_j v\Big|, \Big|\int_{t'}^{t}\Delta_j udS_{j-1}v\Big|\Big\} \leq C_{s,r}|t-t'|^r\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}$$

Putting everything together we therefore arrive at

$$\Big|\int_{t'}^{t} udv\Big| \leq C_{s,r}|t-t'|^r\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}.$$

In particular, if udv is supported in $[-T,T]$, then we have

$$\int_0^t udv \in \mathcal{C}^r \quad \text{and} \quad \Big\|\int_{t'}^{t} udv\Big\|_{\mathcal{C}^r} \leq C_{s,r}(1+T)\|u\|_{\mathcal{C}^r}\|dv\|_{\mathcal{C}^{r-1}}.$$

Now by the continuity of the differential oeprator this also proves first part of (3.3). Uniqueness follow the fact that any distribution with zero derivative must be constant, but the condition at $t=0$ guarantees it to be 0 as well. □

In the special case where v coincides with the trivial function t, we can take r to be equal to s such that

$$\Big\|\int_0^t udt'\Big\|_{\mathcal{C}^s} \leq (1+T)C_s\|u\|_{\mathcal{C}^s}$$

whenever u has compact supported in $[-T,T]$ for some $T>0$. This simpler characterisation will turn out to be a more useful estimate, and will be applied several times throughout this chapter.

Theorem 3.1.2 is now an easy consequence.

Proof of Theorem 3.1.2. It remains to prove the second part of Young's theorem. Using an integration by part, we write

$$T_u v = \sum_{j\geq -1}\sum_{j'<j-1}\int_0^t \Delta_j vd\Delta_{j'}u + \sum_{j\geq -1}\sum_{j'<j-1}\int_0^t \Delta_{j'}ud\Delta_j v,$$

and so

$$\sum_{j\geq -1}\int_0^t S_{j-1}ud\Delta_j v = \sum_{j\geq -1}\int_0^t \Delta_j vdS_{j-1}u + \sum_{j\geq -1}\int_0^t S_{j-1}ud\Delta_j v - \sum_{j\geq -1}\int_0^t \Delta_j vdS_{j-1}u$$
$$= T_u v - \sum_{j\geq -1}\int_0^t \Delta_j vdS_{j-1}u.$$

This gives rise to the decomposition

$$\int_0^t udv = T_u v + \sum_{j\geq -1}\Big(\int_0^t \Delta_j u dS_{j-1}v - \int_0^t \Delta_j v dS_{j-1}u\Big) + \sum_{|j-k|\leq 1}\int_0^t \Delta_j u d\Delta_k v.$$

The required estimate could be done via Proposition 2.3.6 with considerations of the C_b^1 and $\mathcal{C}^{s+r-1}$ norms separately. The boundedness follow from another application of the Fourier multiplier $(i\xi)^{-1}(D)$, from which we obtain

$$\Big\|\sum_{j\geq -1}\int_0^t \Delta_j u dS_{j-1}v\Big\|_{L^\infty} \leq C_{s,r}\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}\sum_{j\geq -1}2^{-j(s+r)},$$
$$\Big\|\sum_{j\geq -1}\int_0^t \Delta_j v dS_{j-1}u\Big\|_{L^\infty} \leq C_{s,r}\|du\|_{\mathcal{C}^{s-1}}\|v\|_{\mathcal{C}^r}\sum_{j\geq -1}2^{-j(s+r)},$$
$$\Big\|\sum_{|j-k|\leq 1}\int_0^t \Delta_j u d\Delta_k v\Big\| \leq C\|u\|_{\mathcal{C}^s}\|dv\|_{\mathcal{C}^{r-1}}\sum_{j\geq -1}2^{-j(s+r)}.$$

What remains of the statement can then be easily proved from Proposition 2.4.1. □

In the case where $s+r\leq 1$, the above estimates fail sharply since by Proposition 2.4.1, the remainder term $R(u,v)$ is no longer guaranteed to be well-defined. Therefore, we cannot expect to solve the differential equation by proving an integral bound. In such cases, we consider a suitable space of signals which can be approximated by regular ones.

Definition 3.1.4. Let s be in $(1/3,1/2)$ and $T>0$. The space of (RDE) enhanceable paths $\mathcal{X}_T^s$[3] consists of all those distributions $\mathbb{X}_T$ in the closure for the image of the map

$$C^\infty([-T,T];\mathbb{R}^n)\to \mathcal{C}^s(\mathbb{R};\mathbb{R}^n)\times\mathcal{C}^{2s-1}(\mathbb{R};\mathbb{R}^n) : X \longmapsto (X, R(X,dX))$$

in $\mathcal{C}^s(\mathbb{R};\mathbb{R}^n)\times\mathcal{C}^{2s-1}(\mathbb{R};\mathbb{R}^n)$.

If we let $I(u,v)$ denote the function in (3.2). Then heuristically, the derivative of $I(u,v)$ can be looked at as u distracted by a noise v on intervals of infinitesimal lengths. Hence one might wonder if it is possible to separate this noise in some reasonable way. By Young's theorem 3.1.2, if we set

$$\mathcal{R}_{u,v}^{I(u,v)} \overset{\text{def}}{=} \sum_{j\geq -1}\Big(\int_0^t \Delta_j u dS_{j-1}v - \int_0^t \Delta_j v dS_{j-1}u\Big) + \sum_{|j-k|\leq 1}\int_0^t \Delta_j u d\Delta_k v$$

Then it is possible to obtain a new expression

$$I(u,v) = T_u v + \mathcal{R}_{u,v}^{I(u,v)}.$$

which behaves more regularly. This motivates the considerations for a space of *para-controlled distributions.*

[3] This corresponds to the space of geometric rough path in the theory of Lyons.

Definition 3.1.5. Let s be in $(1/3, 1/2)$, $r, T > 0$ and $u \in \mathcal{C}^s(\mathbb{R}; \mathbb{R}^d)$. Assume that $\mathbb{X}_T$ is a (RDE) enhancement in $\mathcal{X}_T$. A pair of distributions $(u, v) \in \mathcal{C}^s(\mathbb{R}; \mathbb{R}^d) \times \mathcal{C}^r(\mathbb{R}; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n))$ is *paracontrolled by* $\mathbb{X}_T = (X, R(X, dX))$ if there exists if there exists $\mathbb{X}_T \mathcal{R}^u_v \in \mathcal{C}^{s+r}(\mathbb{R}; \mathbb{R}^d)$ such that

$$u = T_v X + \mathbb{X}_T \mathcal{R}^u_v.$$

We say that v is the *derivative of u paracontrolled by* $\mathbb{X}_T$. The space of all such distributions

$$\mathbb{X}_T \mathcal{D}^r \overset{\text{def}}{=} \left\{ (u, v) \in \mathcal{C}^s(\mathbb{R}; \mathbb{R}^d) \times \mathcal{C}^r(\mathbb{R}; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)) \mid \mathbb{X}_T \mathcal{R}^u_v \overset{\text{def}}{=} u - T_v X \in \mathcal{C}^{s+r}(\mathbb{R}; \mathbb{R}^d) \right\}$$

is equipped with the norm

$$\|(u, v)\|_{\mathbb{X}_T \mathcal{D}^r} \overset{\text{def}}{=} \|v\|_{\mathcal{C}^r(\mathbb{R}; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n))} + \|\mathbb{X}_T \mathcal{R}^u_v\|_{\mathcal{C}^{s+r}(\mathbb{R}; \mathbb{R}^d)}.$$

For the sake of notation, we will not be emphasising the spaces on which the corresponding Hölder-Besov norms are defined so long as there is no ambiguity.

Notice that $\mathbb{X}_T \mathcal{D}^r$ defines a complete normed space. It is clearly a vector space. If $(u_k)_{k \in \mathbb{N}}$ defines a Cauchy sequence in $\mathbb{X}_T \mathcal{D}^r$, then the associated sequence $(v_k, \mathbb{X}_T \mathcal{R}^{u_k}_{v_k})_{k \in \mathbb{N}}$ converges to a unique limit $(v, \mathbb{X}_T \mathcal{R}^u_v)$ in $\mathcal{C}^s \times \mathcal{C}^{2s}$ as k goes to infinity. The result function $u = T_v X + \mathbb{X}_T \mathcal{R}^u_v$ still defines an element of $\mathbb{X}_T \mathcal{D}^r$.

To conclude this section, we note a straightforward application of Theorem 3.1.2 to solve the (RDE) locally for sufficiently regular X.

Corollary 3.1.6. *Let s be in $[1/2, 1)$ and $F \in C^2_b$ such that $\|F\|_{C^1_b}$ is sufficiently small. Then for every $X \in \mathcal{C}^s$ and $\phi \in \mathcal{D}([-2T, 2T])$, there exists a unique global solution to the localised (RDE)*

$$\begin{cases} \dfrac{d}{dt} u = \phi F(u) dX \\ u_{|t=0} = u_0. \end{cases} \tag{3.4}$$

Proof. It is enough to apply Theorem 3.1.2 and conclude that

$$\left\| \int_0^t (F \circ u_1 - F \circ u_2) dX \right\|_{\mathcal{C}^s} \leq C(s, \phi) \|F\|_{C^1_b} \|u_1 - u_2\|_{\mathcal{C}^s} \|X\|_{\mathcal{C}^s}$$

is a contraction for $\|F\|_{C^1_b}$ sufficiently small. □

3.2 The Universal Limit Theorem

Suppose that $T > 0$, the *Ito-Lyons map* is the solution map to the (RDE):

$$\chi^s_T \times \mathbb{R}^n \to \mathcal{C}^s : (\mathbb{X}_T, u_0) \longmapsto u. \tag{3.5}$$

In the theory of rough path, Lyons proved the following *Universal Limit Theorem*

Theorem 3.2.1. *Let s be in $(1/3, 1/2)$ and $T > 0$. Then for every $F \in C^3_b$, the Ito-Lyons map (3.5) can be continuously extended from $\mathcal{D}([-T, T]) \times \mathcal{D}([-T, T]) \times \mathbb{R}^n$. In particular, for every $\mathbb{X}_T \in \mathcal{X}^s_T$, there exists a unique approximate solution in $\mathcal{C}^s$ to the (RDE) (3.1) driven by $\mathbb{X}_T$.*

Note that in view of Proposition 3.1.3, we do not expect the solution to the (RDE) to be contained in $\mathcal{C}^s$ if X does not have compact support. Hence it is in general necessary to consider elements of χ^s_T. This section is devoted to the proof of the above theorem using the harmonic analytic method developed so far.

3.2.1 A Priori Estimates

It would be necessary to first establish some technical estimates which will allow us to obtain concrete bounds for solutions of the form $u = \phi T_{F\circ u}X + \mathcal{R}^{u,X}_{F\circ u,T}$, where ϕ is in $\mathcal{D}([-2T, 2T])$ and equal to 1 on $[-T,T]$, and u solves the localised (RDE) (3.4) weighted by ϕ. In the regular case, if X is supported in $[-T,T]$, then in addition u coincides with the original (RDE) on $[-T,T]$.

From what we have seen with Young's theorem, it would be convenient to avoid dealing directly with $R(F\circ u, X)$. The following statement provides a reasonable bound in terms of $R(X, dX)$.

Lemma 3.2.2. *Let s be in $(1/3, 1/2)$ and $F \in C^3_b$. If $T > 0$, then for every solution to the localised (RDE) driven by X, there exists constant $C(s,\phi)$ such that*

$$\|R(F\circ u, dX)\|_{\mathcal{C}^{2s-1}}$$
$$\leq C(s,\phi)(\|F\|_{C^2_b} + \|F\|^2_{C^2_b})(\|u\|_{\mathcal{C}^s} + \|\mathcal{R}^{u,T}_{F\circ u,X}\|_{\mathcal{C}^{2s}})(\|X\|_{\mathcal{C}^s} + \|X\|^2_{\mathcal{C}^s} + \|R(X,dX)\|_{\mathcal{C}^s}).$$

Proof. Using a paralinearisation and the paracontrolled structure, we calculate

$$F\circ u = T_{F\circ u}(\phi T_{F\circ u}X) + T_{F'\circ u}\mathcal{R}^{u,T}_{F\circ u,X} + R_F u.$$

Applying twice the commutator 2.4.10, we then have

$$\begin{aligned} R(F\circ u, dX) &= R(T_{F'\circ u}\mathcal{R}^{u,T}_{F\circ u,X}, dX) + \phi(F'\circ u)(F\circ u)R(X,dX) \\ &\quad + \phi(F'\circ u)(F\circ u)R(X,dX) + \phi(F'\circ u)\Gamma(F\circ u, X, dX) + \Gamma(F'\circ u, \phi T_{F\circ u}X, dX) \\ &\quad + (F'\circ u)(T_{F\circ u}X)R(\phi, dX) + (F'\circ u)\Sigma(\phi, T_{F\circ u}X, dX) + R(R_F u, dX). \end{aligned}$$

By making uses of Lemma 2.4.1, 2.4.10 and Proposition 2.4.5, components of the above sum can be estimated as follows

$$\|R(T_{F'\circ u}\mathcal{R}^{u,T}_{F\circ u,X}, dX)\|_{\mathcal{C}^{2s-1}} \leq C_s\|F\|_{C^3_b}\|\mathcal{R}^{u,T}_{F\circ u,X}\|_{\mathcal{C}^{2s}}\|u\|_{\mathcal{C}^s}\|X\|_{\mathcal{C}^s}$$

$$\|\phi(F'\circ u)(F\circ u)R(X,dX)\|_{\mathcal{C}^{2s-1}} \leq C_s\|F\|^2_{C^3_b}\|\phi\|_{C^1_b}\|u\|_{\mathcal{C}^s}\|R(X,dX)\|_{\mathcal{C}^{2s}},$$

$$\|\phi R(R_F u, dX)\|_{\mathcal{C}^{2s-1}} \leq C_s\|F\|_{C^3_b}\|\phi\|_{C^3_b}(1 + \|F\|_{{}_L C^3_b}\|X\|_{\mathcal{C}^s})(\|u\|_{\mathcal{C}^s} + \|\mathcal{R}^u_{F\circ u,X}\|_{\mathcal{C}^{2s}})\|X\|_{\mathcal{C}^s}$$

On the other hand, the quantities

$$\|\phi F'\circ u\Gamma(F\circ u, X, dX)\|_{\mathcal{C}^{2s-1}}, \qquad \|\Gamma(F'\circ u, \phi T_{F\circ u}X, dX)\|_{\mathcal{C}^{2s-1}},$$
$$\|(F'\circ u)\Sigma(\phi, T_{F\circ u}X, dX)\|_{\mathcal{C}^{2s-1}}, \qquad \|(F'\circ u)(T_{F\circ u}X)R(\phi, dX)\|_{\mathcal{C}^{2s-1}}.$$

can be easily estimated by $C_s\|\phi\|_{C^2_b}\|F\|^2_{C^3_b}\|X\|^2_{\mathcal{C}^s}\|u\|_{\mathcal{C}^s}$. The required bound is an immediate consequence of these estimates. □

Our main objective of this subsection is to obtain a local bound for solution to the localised (RDE). Clearly, this will require a bound on $\mathcal{R}^{u,T}_{F\circ u,X}$.

Lemma 3.2.3. *Let s be in $(1/3, 1/2)$ and $F \in C_b^3$ be such that $\|F\|_{C_b^3}$ is sufficiently small. If $T > 0$, then for every solution to the localised (RDE) driven by X, there exists constant $C(s,\phi)$ such that*

$$\left\|\frac{d}{dt}\mathcal{R}^{u,T}_{F\circ u,X}\right\|_{\mathcal{C}^{2s-1}} \leq C(s,\phi)(1+T)(\|F\|_{C_b^3} + \|F\|^2_{C_b^3}) \\ \times (\|F\|_{C_b^3}\|X\|_{\mathcal{C}^s} + \|u\|_{\mathcal{C}^s})(\|X\|_{\mathcal{C}^s} + \|X\|^2_{\mathcal{C}^s} + \|R(X,dX)\|_{\mathcal{C}^s}).$$

Proof. Using the structure of solution, we have

$$\frac{d}{dt}\mathcal{R}^{u,T}_{F\circ u,X} = \phi T_{dX}F\circ u - \phi' T_{F\circ u}X - \phi T_{(F\circ u)'}X + \phi R(F\circ u, dX).$$

Proposition 2.4.1 ensures that

$$\max\left\{\|\phi T_{dX}F\circ u\|_{\mathcal{C}^{s-1}},\ \|\phi' T_{F\circ u}X\|_{\mathcal{C}^{s-1}}, \|\phi T_{((F\circ u)'}X\|_{\mathcal{C}^{2s-1}}\right\} \\ \leq C_s\|\phi\|_{C_b^2}\|F\|_{C_b^3}\|X\|_{\mathcal{C}^s}\|u\|_{\mathcal{C}^s}.$$

So Lemma 3.2.2 gives

$$\left\|\frac{d}{dt}\mathcal{R}^{u,T}_{F\circ u,X}\right\|_{\mathcal{C}^{2s-1}} \leq C(s,\phi)(1+T)(\|F\|_{C_b^3} + \|F\|^2_{C_b^3}) \\ \times (\|u\|_{\mathcal{C}^s} + \|\mathcal{R}^{u}_{F\circ u,X}\|_{\mathcal{C}^{2s}})(\|X\|_{\mathcal{C}^s} + \|X\|^2_{\mathcal{C}^s} + \|R(X,dX)\|_{\mathcal{C}^s}).$$

Note that by uniqueness in Proposition 3.1.3 we have

$$\mathcal{R}^{u,T}_{F\circ u,X} = u_0 - (T_{F\circ u}X(0) + \int_0^t \frac{d}{dt}\mathcal{R}^{u,T}_{F\circ u,X}dt', \quad \text{and}$$

$$\|\mathcal{R}^{u,T}_{F\circ u,X}\|_{\mathcal{C}^{2s}} \leq (1+T)\Big(\Big(\|F\|_{C_b^3}\|X\|_{\mathcal{C}^s} + \|u\|_{\mathcal{C}^s}\Big) + \left\|\frac{d}{dt}\mathcal{R}^{u,T}_{F\circ u,X}\right\|_{\mathcal{C}^{2s-1}}\Big).$$

By further estimating the derivative of the remainder term with the inequality before, the required estimate follows for $\|F\|_{C_b^3}$ sufficiently small. □

Proposition 3.2.4. *Let s be in $(1/3, 1/2)$ and $F \in C_b^3$ be such that $\|F\|_{C_b^3}$ is sufficiently small. If $T > 0$, then for every solution to the localised (RDE) driven by X, there exists constant $C(s,\phi)$ such that*

$$\|u\|_{\mathcal{C}^s} \leq (1+T^2)C_s\Big(|u_0| + (\|F\|_{C_b^3} + \|F\|^2_{C_b^3}) \\ \times (\|X\|_{\mathcal{C}^s} + 1)(\|X\|_{\mathcal{C}^s} + \|X\|^2_{\mathcal{C}^s} + \|R(X,dX)\|_{\mathcal{C}^{2s-1}})\Big).$$

Proof. By Proposition 3.1.3, we have

$$u = u_0 + \int_0^t \frac{d}{dt'}u dt'.$$

Therefore by Proposition 3.1.3

$$\begin{aligned}\|u\|_{\mathcal{C}^s} &\leq (1+T)C_s\Big(|u_0| + \left\|\frac{d}{dt}u\right\|_{\mathcal{C}^{s-1}}\Big) \\ &\leq (1+T)C_s\Big(|u_0| + \|\phi T_{(F\circ u)'}X\|_{\mathcal{C}^{s-1}} + \|\phi' T_{F\circ u}X\|_{\mathcal{C}^{s-1}} \\ &\qquad + \|\phi T_{F\circ u}dX\|_{\mathcal{C}^{s-1}} + \left\|\frac{d}{dt}\mathcal{R}^{u,T}_{F\circ u,X}\right\|_{\mathcal{C}^{2s-1}}\Big).\end{aligned} \tag{3.6}$$

It suffices to estimate terms in expression (3.6). Proposition 2.4.1 ensures that

$$\|\phi T_{F\circ u} dX\|_{\mathcal{C}^{s-1}} + \|\phi T_{dX} F\circ u\|_{\mathcal{C}^{2s-1}} + \|\phi' T_{F\circ u} X\|_{\mathcal{C}^{2s-1}} \leq C_s \|\phi\|_{C_b^2} \|F\|_{C_b^3} \|X\|_{\mathcal{C}^s} \|u\|_{\mathcal{C}^s}.$$

Therefore Lemma 3.2.3 gives

$$\begin{aligned} \|u\|_{\mathcal{C}^s} \leq & C(s,\phi)(1+T^2)\Big(|u_0| + (\|F\|_{C_b^3} + \|F\|^2_{C_b^3}) \\ & \times (\|F\|_{C_b^3}\|X\|_{\mathcal{C}^s} + \|u\|_{\mathcal{C}^s})(\|X\|_{\mathcal{C}^s} + \|X\|^2_{\mathcal{C}^s} + \|R(X,dX)\|_{\mathcal{C}^s}). \end{aligned}$$

For $\|F\|_{C_b^3}$ sufficiently small the desired estimate follows. □

Remark 3.2.5. In the above $\|F\|_{C_b^3}$ only depends on elements of the signal and its localisation $(X, R(X,dX), \phi)$.

3.2.2 Lipschitz Continuity of the Solution Map

In this subsection we will establish the locally Lipschitz continuity of the Ito-Lyons map to the localised (RDE). This will be a critical result needed for proving convergences for our solutions to our original (RDE).

Proposition 3.2.6. *Let s be in $(1/3, 1/2)$ and $F \in \mathcal{C}_b^3$ such that $\|F\|_{C_b^3}$ is sufficiently small. If $T > 0$, then the Ito-Lyons map to the localised (RDE)*

$$\mathcal{C}^s \times \mathcal{C}^{2s-1} \times \mathbb{R}^n \to \mathcal{C}^{s-1} : (X, R(X,dX), u_0) \longrightarrow u$$

is locally Lipschitz continuous.

Proof. Since the proof is going to be of length, we will separate the argument into three steps:

- First, we will establish the critical quantities which need to be estimated.
- Second, we will estimate these quantities respectively.
- Finally, by injecting these estimate back into what needs to be bounded, we will conclude the proof of the claim.

First Step: Setting Up the Estimates

Suppose u_1, u_2 are two solutions to the (RDE) driven by X_1, X_2, with initial data $u_{1,0}, u_{2,0}$ respectively. By uniqueness of the definite integral (3.2), we have

$$u_1 - u_2 = u_{1,0} - u_{2,0} + \int_0^t \frac{d}{dt'}(u_1 - u_2)dt'.$$

Applying Corollary 3.1.3, we then have

$$\begin{aligned} \|u_1 - u_2\|_{\mathcal{C}^s} &\leq (1+T)C_s\Big(|u_{1,0} - u_{2,0}| + \Big\|\frac{d}{dt}(u_1-u_2)\Big\|_{\mathcal{C}^{s-1}}\Big) \\ &\leq (1+T)C_s\Big(|u_{1,0} - u_{2,0}| + \Big\|\frac{d}{dt}\phi T_{F\circ u} X_1 - \frac{d}{dt}\phi T_{F\circ u_2} X_2\Big\|_{\mathcal{C}^{s-1}} \\ &\qquad + \Big\|\frac{d}{dt}\mathcal{R}^{u_1,T}_{F\circ u_1, X_1} - \frac{d}{dt}\mathcal{R}^{u_2,T}_{F\circ u_2, X_2}\Big\|_{\mathcal{C}^{s-1}}\Big). \end{aligned} \tag{3.7}$$

It is easy to see that

$$\begin{aligned}\Big\|\frac{d}{dt}T_{F\circ u_1}X_1-\frac{d}{dt}T_{F\circ u_2}X_2\Big\|_{\mathcal{C}^{s-1}}&\le C_s\Big(\Big\|T_{(F\circ u_1)'}(X_1-X_2)\Big\|_{\mathcal{C}^{s-1}}\\&+\Big\|T_{(F\circ u_1-F\circ u_2)'}X_2\Big\|_{\mathcal{C}^{s-1}}+\Big\|T_{F\circ u_1}(X_1-X_2)\Big\|_{\mathcal{C}^{s-1}}+\Big\|T_{F\circ u_1-F\circ u_2}X_2\Big\|_{\mathcal{C}^{s-1}}\Big),\end{aligned}\tag{3.8}$$

so it remains to consider the differences of the derivatives of $\mathcal{R}^{u_1,T}_{F\circ u_1,X_1}-\mathcal{R}^{u_2,T}_{F\circ u_2,X_2}$. For $j=1,2$, recall that we have

$$\frac{d}{dt}\mathcal{R}^{u_j,T}_{F\circ u_j,X_j}=\phi T_{dX_j}F\circ u_j-\phi' T_{F\circ u_j}X_j-\phi T_{(F\circ u_j)'}X_j+\phi R(F\circ u_j,dX_j),$$

where the remainder term has the representation

$$\begin{aligned}R(F\circ u_j,dX_j)&=F'\circ u_jR(u_j,dX_j)+\Pi_F(u_j,dx_j)\\&=F'\circ u_jR(\phi T_{F\circ u_j}X_j,dX_j)+F'\circ u_jR(\mathcal{R}^{u_j,T}_{F\circ u_j,X_j},dX_j)+\Pi_F(u_j,dX_j)\end{aligned}$$

and furthermore

$$\begin{aligned}R(\phi T_{F\circ u_j}X_j,dX_j)&=\phi R(T_{F\circ u_j}X_j,dX_j)+(T_{F\circ u_j}X_j)R(\phi,dX_j)+\Sigma(\phi,T_{F\circ u_j}X_j,dX_j)\\&=\phi(F\circ u_j)R(X_j,dX_j)+\phi\Gamma(F\circ u_j,X_j,dX_j)+(T_{F\circ u_j}X_j)R(\phi,dX_j)\\&\quad+\Sigma(\phi,T_{F\circ u_j}X_j,dX_j).\end{aligned}$$

Therefore, setting

$$\begin{aligned}D^{(1)}&\overset{\text{def}}{=}T_{dX_1}F\circ u_1-T_{dX_2}F\circ u_2,\qquad D^{(2)}\overset{\text{def}}{=}T_{(F\circ u_1)'}X_1-T_{(F\circ u_2)'}X_2,\\D^{(3)}&\overset{\text{def}}{=}\phi(F'\circ u_1)(F\circ u_1)R(X_1,dX_1)-\phi(F'\circ u_2)(F\circ u_2)R(X_2,dX_2),\\D^{(4)}&\overset{\text{def}}{=}\phi(F'\circ u_1)\Gamma(F\circ u_1,X_1,dX_1)-\phi(F'\circ u_2)\Gamma(F\circ u_2,X_2,dX_2),\\D^{(5)}&\overset{\text{def}}{=}(F'\circ u_1)(T_{F\circ u_1}X_1)R(\phi,dX_1)-(F'\circ u_2)(T_{F\circ u_2}X_2)R(\phi,dX_2),\\D^{(6)}&\overset{\text{def}}{=}(F'\circ u_1)\Sigma(\phi,T_{F\circ u_1}X_1,dX_1)-(F'\circ u_2)\Sigma(\phi,T_{F\circ u_2}X_2,dX_2),\\D^{(7)}&\overset{\text{def}}{=}(F'\circ u_1)R(\mathcal{R}^{u_1}_{F\circ u_1,X_1},dX_1)-(F'\circ u_2)R(\mathcal{R}^{u_2}_{F\circ u_2,X_2},dX_2),\\D^{(8)}&\overset{\text{def}}{=}\Pi_F(u_1,dX_1)-\Pi_F(u_2,dX_2),\end{aligned}$$

yields the expression

$$\frac{d}{dt}\mathcal{R}^{u_1,T}_{F\circ u_1,X_1}-\frac{d}{dt}\mathcal{R}^{u_2,T}_{F\circ u_2,X_2}=\phi'(T_{F\circ u_1}X_1-T_{F\circ u_2}X_2)+\phi\sum_{j=1}^{8}D^{(j)}\tag{3.9}$$

To prove the claim, it suffices to estimate the differences in (3.8) and (3.9).

Second Step: Estimation of the Differences

Let us now estimate the required quantities. Because we have

$$\begin{aligned}\|D^{(1)}\|_{\mathcal{C}^{2s-1}}&\le\|T_{dX_1-dX_2}F\circ u_1\|_{\mathcal{C}^{2s-1}}+\|T_{dX_2}(F\circ u_1-F\circ u_2)\|_{\mathcal{C}^{2s-1}},\\\|D^{(2)}\|_{\mathcal{C}^{2s-1}}&\le\|T_{(F\circ u_1)'-(F\circ u_2)'}X_1\|_{\mathcal{C}^{2s-1}}+\|T_{(F\circ u_2)'}(X_1-X_2)\|_{\mathcal{C}^{2s-1}},\end{aligned}$$

and

$$\|T_{F\circ u_1}X_1 - T_{F\circ u_2}X_2\|_{\mathcal{C}^{2s-1}} \le \|T_{F\circ u_1}(X_1 - X_2)\|_{\mathcal{C}^{2s-1}} + \|T_{F\circ u_1 - F\circ u_2}X_2\|_{\mathcal{C}^{2s-1}}. \quad (3.10)$$

The differences involving paraproducts are easily bounded by Proposition 2.4.1 with the Lipschitz continuity of F, and the difference involving the commutator can be directly estimated by Lemma 2.4.11,

$$\|D^{(1)}\|_{\mathcal{C}^{2s-1}} \le C_s\|F\|_{C_b^3}(1 + \|u_1\|_{\mathcal{C}^s})(1 + \|X_2\|_{\mathcal{C}^s})(\|X_1 - X_2\|_{\mathcal{C}^s} + \|u_1 - u_2\|_{\mathcal{C}^s})$$
$$\|D^{(2)}\|_{\mathcal{C}^{2s-1}} \le C_s\|F\|_{C_b^3}(1 + \|u_2\|_{\mathcal{C}^s})(1 + \|X_1\|_{\mathcal{C}^s})(\|u_1 - u_2\|_{\mathcal{C}^s} + \|X_1 - X_2\|_{\mathcal{C}^s})$$
$$\|D^{(8)}\|_{\mathcal{C}^{2s-1}} \le C_s\|F\|_{C_b^3}\Big(1 + (\|u_1\|_{\mathcal{C}^s} + \|u_2\|_{\mathcal{C}^s})^2 + \|X_2\|_{\mathcal{C}^s}\Big)(\|u_1 - u_2\|_{\mathcal{C}^s} + \|X_1 - X_2\|_{\mathcal{C}^s}).$$

The third difference in (3.9) can be estimated as follows

$$\|D^{(3)}\|_{\mathcal{C}^{2s-1}} \le \|\phi\|_{C_b^2}\|\Big((F'\circ u_1)(F\circ u_1) - (F'\circ u_2)(F\circ u_2)\Big)R(X_1, dX_1)\|_{\mathcal{C}^{2s-1}}$$
$$+ \|\phi\|_{C_b^2}\|(F'\circ u_2)(F\circ u_2)(R(X_1, dX_1) - R(X_2, dX_2))\|_{\mathcal{C}^{2s-1}}.$$

We then use that

$$\|\Big((F'\circ u_1)(F\circ u_1) - (F'\circ u_2)(F\circ u_2)R(X_1, dX_1)\Big)R(X_1, dX_1)\|_{\mathcal{C}^{2s-1}}$$
$$\le \Big(\|(F'\circ u_1 - F'\circ u_2)F\circ u_1\|_{\mathcal{C}^s} + \|F'\circ u_2(F\circ u_1 - F\circ u_2)\|_{\mathcal{C}^s}\Big)\|R(X_1, dX_1)\|_{\mathcal{C}^{2s-1}}$$

to obtain the bound

$$\|D^{(3)}\|_{\mathcal{C}^{2s-1}} \le C_s\|\phi\|_{C_b^2}\|F\|^2_{C_b^3}(1 + \|u_1\|_{\mathcal{C}^s})(1 + \|u_2\|_{\mathcal{C}^s})(1 + \|R(X_1, dX_1)$$
$$\times \|_{\mathcal{C}^{2s-1}})(\|u_1 - u_2\|_{\mathcal{C}^s} + \|R(X_1, dX_1) - R(X_2, dX_2)\|_{\mathcal{C}^{2s-1}}).$$

Analogously, for the fourth difference, we bound by Lemma 2.4.10 that

$$\|D^{(4)}\|_{\mathcal{C}^{2s-1}} \le C_s\|\phi\|_{C_b^2}\|F\|^2_{C_b^3}(\|X_1 - X_2\|_{\mathcal{C}^s} + \|u_1 - u_2\|_{\mathcal{C}^s})(1 + \|u_2\|_{\mathcal{C}^s})$$
$$\times (1 + \|X_1\|_{\mathcal{C}^s} + \|X_1\|^2_{\mathcal{C}^s})(1 + \|X_2\|^2_{\mathcal{C}^s} + +\|X_2\|_{\mathcal{C}^s}).$$

The estimation of the fifth difference requires the consideration of

$$\|D^{(5)}\|_{\mathcal{C}^{2s-1}} \le \|F'\circ u_1 T_{F\circ u_1}X_1 R(\phi, dX_1 - dX_2)\|_{\mathcal{C}^{2s-1}}$$
$$+ \|(F'\circ u_1 T_{F\circ u_1}X_1 - F'\circ u_2 T_{F\circ u_2}X_s)R(\phi, dX_2)\|_{\mathcal{C}^{2s-1}}.$$

Because

$$\|F'\circ u_1 T_{F\circ u_1}X_1 - F'\circ u_2 T_{F\circ u_2}X_2\|_{\mathcal{C}^s} \le C_s\|F\|^2_{C_b^3}(1 + \|u_1\|_{\mathcal{C}^s})(1 + \|u_2\|_{\mathcal{C}^s})$$
$$\times (1 + \|X_1\|_{\mathcal{C}^s})(1 + \|X_2\|_{\mathcal{C}^s})(\|u_1 - u_2\|_{\mathcal{C}^s} + \|X_1 - X_2\|_{\mathcal{C}^s}),$$

the estimate

$$\|D^{(5)}\|_{\mathcal{C}^{2s-1}} \le C_s\|F\|^2_{C_b^3}\|\phi\|_{C_b^2}(1 + \|u_1\|_{\mathcal{C}^s})(1 + \|u_2\|_{\mathcal{C}^s})(1 + \|X_1\|_{\mathcal{C}^s})$$
$$\times (1 + \|X_2\|_{\mathcal{C}^s})(\|u_1 - u_2\|_{\mathcal{C}^s} - \|X_1 - X_2\|_{\mathcal{C}^s})$$

follows. Continuing our calculation

$$\|D^{(6)}\|_{\mathcal{C}^{2s-1}} \leq C_s\|F\|^2_{C^3_b}\|\phi\|_{C^2_b}(1+\|u_1\|_{\mathcal{C}^s}+\|u_2\|_{\mathcal{C}^s})(\|X_2\|_{\mathcal{C}^s}+\|X_2\|^2_{\mathcal{C}^s})(1+\|X_1\|_{\mathcal{C}^s}) \times (\|X_1-X_2\|_{\mathcal{C}^s}+\|u_1-u_2\|_{\mathcal{C}^s}).$$

And finally for the seventh difference

$$\|D^{(7)}\|_{\mathcal{C}^{2s-1}} \leq \|F\|_{C^3_b}C_s(\|\mathcal{R}^{u_1}_{Fou_1,X_1}-\mathcal{R}^{u_2}_{Fou_2,X_2}\|_{\mathcal{C}^{2s}}+\|X_1-X_2\|_{\mathcal{C}^s}+\|u_1-u_2\|_{\mathcal{C}^s}) \times (1+\|X_1\|_{\mathcal{C}^s})(1+\|\mathcal{R}^{u_2}_{Fou_2,X_2}\|_{\mathcal{C}^{2s}}).$$

These give a complete description of the differences.

Final Step: Substitution and Conclusion

It remains to reinvestigate the difference (3.7). Putting everything together, we see that (3.8) satisfies

$$\left\|\frac{d}{dt}T_{Fou_1}X_1-\frac{d}{dt}T_{Fou_2}X_2\right\|_{\mathcal{C}^{2s-1}} \leq C_s\|F\|_{C^3_b}(1+\|u_1\|_{\mathcal{C}^s})(1+\|X_2\|_{\mathcal{C}^s})(\|X_1-X_2\|_{\mathcal{C}^s}+\|u_1-u_2\|_{\mathcal{C}^s}).$$

On the other hand, injecting the previous estimates into (3.9) and apply Lemma 3.2.3, Proposition 3.2.4 for $\|F\|_{C^3_b}$ sufficient small reveals that there exists positive constant $C(s,u,\mathcal{R}^{u,T}_{Fou,X},\phi)$, which is locally bounded in terms of X and $R(X,dX)$, such that

$$\left\|\frac{d}{dt}\mathcal{R}^{u_1,T}_{Fou_1,X_1}-\frac{d}{dt}\mathcal{R}^{u_2,T}_{Fou_2,X_2}\right\|_{\mathcal{C}^{2s-1}} \leq C(s,u,\mathcal{R}^{u,T}_{Fou,X},\phi)(\|F\|_{C^3_b}+\|F\|^2_{C^3_b})\prod_{j=1,2}(1+\|X_j\|_{\mathcal{C}^s}+\|X_j\|^2_{\mathcal{C}^s}+\|R(X_j,dX_j)\|_{\mathcal{C}^{2s-1}}) \times (\|X_1-X_2\|_{\mathcal{C}^s}+\|\mathcal{R}^{u_1,T}_{Fou_1,X_1}-\mathcal{R}^{u_2,T}_{Fou_2,X_2}\|_{\mathcal{C}^{2s}}+\|R(X_1,dX_1)-R(X_2,dX_2)\|_{\mathcal{C}^{2s-1}}).$$

Therefore, if one uses the fact that

$$\|\mathcal{R}^{u_1,T}_{Fou_1,X_1}-\mathcal{R}^{u_2,T}_{Fou_2,X_2}\|_{\mathcal{C}^{2s}} \leq (1+T)\Big(|u_{1,0}-u_{2,0}|+\|T_{Fou_1}X_1-T_{Fou_2}X_2\|_{L^\infty}+\left\|\frac{d}{dt}\mathcal{R}^{u_1,T}_{Fou_1,X_1}-\frac{d}{dt}\mathcal{R}^{u_2,T}_{Fou_2,X_2}\right\|_{\mathcal{C}^{2s-1}}\Big), \tag{3.11}$$

then for $\|F\|_{C^3_b}$ sufficiently small, we obtain our desired estimate of (3.7) as follows

$$\|u_1-u_2\|_{\mathcal{C}^s} \leq (1+T^2)C(s,u,X,\mathcal{R}^{u,T}_{Fou,X},\phi) \times \Big(|u_{1,0}-u_{2,0}|+(\|X_1-X_2\|_{\mathcal{C}^s}+\|R(X_1,dX_1)-R(X_2,dX_2)\|_{\mathcal{C}^{2s-1}})\Big).$$

for a constant $C(s,u,X,\mathcal{R}^{u,T}_{Fou,X},\phi)$ which is locally bounded by in terms of X and $R(X,dX)$. □

3.2.3 Approximate Solution in Besov Spaces and Proof of the Theorem

With the result proven above at hand, we can now establish the existence of a unique global solution to the (RDE) driven by signals with compact support. In fact, we will prove a somewhat stronger statement. Let $\mathcal{C}^s_{loc}$ denotes the space of tempered distributions such that $\phi u \in \mathcal{C}^s$ for every $\phi \in \mathcal{D}$, in which a sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathcal{C}^s_{loc}$ if as n goes in infinity if and only if $\lim_{n\to\infty} \|\phi(u-u_n)\|_{\mathcal{C}^s} = 0$ for all $\phi \in \mathcal{D}$. We have the following result.

Proposition 3.2.7. *Let s be in $(1/3, 1/2)$ and $F \in C^3_b$. Assume that $(u^\epsilon)_{\epsilon>0}$ is a family of elements in $\mathcal{C}^s$ which solves the (RDE) (3.1) driven by $(X^\epsilon)_{\epsilon>0}$ with initial data $(u^\epsilon_0)_{\epsilon>0}$. If $\lim_{\epsilon\to 0}(X^\epsilon, R(X^\epsilon, dX^\epsilon), u^\epsilon_0)$ exists in $\mathcal{C}^s \times \mathcal{C}^{2s-1} \times \mathbb{R}^n$, then u^ϵ converges in $\mathcal{C}^s_{loc}$ as ϵ goes to 0. The limit u does not depend on the choice of approximating families $(X^\epsilon, R(X^\epsilon, dX^\epsilon), u^\epsilon_0)$.*

Proof. *A priori*, we only have existence of approximate solution to the localised (RDE) (3.4) for $\|F\|_{C^3_b}$ sufficiently small. Recall the scaling operator Λ_λ defined for $\lambda > 0$ and set

$$u^\lambda \stackrel{\text{def}}{=} \Lambda_\lambda u, \quad \text{and} \quad X^\lambda \stackrel{\text{def}}{=} \lambda^{-s}\Lambda_\lambda X,$$

where u solves the (RDE) driven by X. Then u^λ solves the scaled equation

$$\begin{cases} \dfrac{d}{dt}u^\lambda = \lambda^s F(u^\lambda) dX^\lambda, \\ u^\lambda{}_{|t=0} = u_0. \end{cases}$$

By Lemma 2.4.7, for $\lambda < 1$ this scaling is chosen such that

$$\|dX^\lambda\|_{\mathcal{C}^{s-1}} \le C_s \|dX\|_{\mathcal{C}^s}, \quad \text{and}$$
$$\|R(X^\lambda, dX^\lambda)\|_{\mathcal{C}^{2s-1}} \le C_s\Big(\|R(X, dX)\|_{\mathcal{C}^{2s-1}} + \|X\|_{\mathcal{C}^s}\|dX\|_{\mathcal{C}^s}\Big).$$

Thus, for every $\phi \in \mathcal{D}(\mathbb{R})$ there exists an appropriately small λ such that there exists an unique global solution to

$$\begin{cases} \dfrac{d}{dt}\tilde{u}^\lambda = \lambda^s \phi F(\tilde{u}^\lambda) dX^\lambda, \\ \tilde{u}^\lambda_{|t=0} = u_0. \end{cases} \tag{3.12}$$

Equivalently, releasing the scaling reveals that $\tilde{u} \stackrel{\text{def}}{=} \Lambda_{\lambda^{-1}}\tilde{u}^\lambda$ is the unique global solution to

$$\begin{cases} \dfrac{d}{dt}\tilde{u} = \phi_\lambda F(\tilde{u}) dX, \quad \phi_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda^{-1}}\phi \\ \tilde{u}_{|t=0} = u_0. \end{cases} \tag{3.13}$$

In particular, if ϕ is chosen to be identically equal to 1 on $[-1, 1]$. Then the solutions to (3.13) and (3.1) agree on $[-\lambda, \lambda]$. Now by Proposition 3.2.6 and 2.3.4, the solutions $(\tilde{u}^\epsilon)_{\epsilon>0}$ to the regularised (RDE) (3.13) driven by $(X^\epsilon)_{\epsilon>0}$ with initial conditions $(u^\epsilon_0)_{\epsilon>0}$

converges to a unique limit in $\mathcal{C}^s$ as ϵ goes to 0. But by our previous comment, $(\tilde{u}^\epsilon)_{\epsilon>0}$ and $(u^\epsilon)_{\epsilon>0}$ coincide on $[-\lambda, \lambda]$. Thus if we choose u to be such that u and $\tilde{u}$ coincide on $[-\lambda, \lambda]$, then $\lim_{\epsilon\to 0} \|\psi(u - u^\epsilon)\|_{\mathcal{C}^s} = 0$ for every $\psi \in \mathcal{D}([-\lambda, \lambda])$. Because the above construction was independent of the initial conditions $(u_0^\epsilon)_{\epsilon>0}$, we can iterate this procedure on any interval of length 2λ. Moreover, if $\psi \in \mathcal{D}$ is arbitrary, then it can be written as a finite sum of smooth functions with support contained in intervals of such lengths, so that by what was proved we have $\lim_{\epsilon\to 0} \psi u^\epsilon = \psi u$ in $\mathcal{C}^s$. This concludes the claim. □

Notice that by the scaling above argument, if X is regular, then the (RDE) driven by X in fact admits solution on every compact interval, independent of the size of $\|F\|_{C_b^3}$. Using this fact, we can finally prove Lyon's theorem.

Proof of Theorem 3.2.1. By definition, for every $\mathbb{X}_T \in \mathcal{X}_T^s$ there exists a family of regularised signals $(X^\epsilon)_{\epsilon>0}$ in $\mathcal{D}([-T, T])$ such that $\lim_{\epsilon\to 0} X^\epsilon = X$ in $\mathcal{C}^s$. Let $(u^\epsilon)_{\epsilon>0}$ be solutions to the (RDE) driven by $(X^\epsilon)_{\epsilon>0}$ on $[-T, T]$. Take $\phi \in \mathcal{D}$ such that $[-T, T] \subseteq \operatorname{Supp} \phi$ and $\phi = 1$ near $[-T, T]$, then by the above we have that the function $u = \lim_{\epsilon\to 0} u^\epsilon$ in $\mathcal{C}^s$ solves the (RDE) on $[-T, T]$. Since the derivative of u has compact support, u is constant on $[-T, T]^c$. Therefore letting u flow starting from the initial condition x_0 to time T determines uniquely this constant. Extending u to this constant outside of $[-T, T]$ then yields the unique solution in $\mathcal{C}^s$. □

3.2.4 Interpretation of the Product

With some efforts we have successfully constructed a unique function in $\mathcal{C}^s$, which solves the (RDE) (3.1) driven by signals $\mathbb{X}_T \in \mathcal{X}_T^s$ in the sense of the Universal Limit Theorem 3.2.1. Given the special structure of that solution, we could in fact interpret it appropriately so that it satisfy the corresponding (RDE). Indeed, expressing the paracontrolled structure $u = T_{F\circ u} dX + \mathbb{X}_T \mathcal{R}^u_{F\circ u}$, we have

$$F(u)dX = T_{F\circ u} dX + T_{dX} F \circ u + (F' \circ u)\Gamma(F \circ u, X, dX) \\ + (F' \circ u)(F \circ u)R(X, dX) + (F' \circ u)R(\mathbb{X}_T \mathcal{R}^u_{F\circ u}, dX) + \Pi_F(u, dX).$$

The only terms which is a priori not well defined is $(F' \circ u)(F \circ u)R(X, dX)$ and $(F' \circ u)R(\mathbb{X}_T \mathcal{R}^u_{F\circ u}, dX)$, but we formally have the estimates

$$\|(F' \circ u)(F \circ u)R(X, dX)\|_{\mathcal{C}^{s-1}} \leq C_s \|F\|^2_{C_b^3} \|u\|_{\mathcal{C}^s} \|R(X, dX)\|_{\mathcal{C}^{2s}}, \quad \text{and}$$
$$\|(F' \circ u)R(\mathbb{X}_T \mathcal{R}^u_{F\circ u}, dX)\|_{\mathcal{C}^{s-1}} \leq C_s \|F\|_{C_b^3} \|u\|_{\mathcal{C}^s} \|\mathbb{X}_T \mathcal{R}^u_{F\circ u}\|_{\mathcal{C}^{2s}} \|X\|_{\mathcal{C}^s},$$

and therefore the product is well-defined element of $\mathcal{C}^{s-1}$ as soon as we have $\mathbb{X}_T \mathcal{R}^u_{F\circ u} \in \mathcal{C}^{2s}$ and $\mathbb{X}_T$ is a (RDE) enhancement.

For more general situation, we look at following proposition.

Proposition 3.2.8. *Let s be in $(1/3, 1)$ and $F \in C_b^2$. If $T > 0$ and $\mathbb{X}_T \in \mathcal{X}_T^s$ is a (RDE) enhancement. Then the map*

$$\begin{cases} \mathbb{X}_T \mathcal{D}^s \to \mathcal{C}^{s-1} \\ (u, v) \mapsto T_{F\circ u} dX + T_{dX} F \circ u + (F' \circ u)\Gamma(v, X, dX) \\ \qquad (F' \circ u)vR(X, dX) + (F' \circ u)R(\mathbb{X}_T \mathcal{R}^u_v, dX) + \Pi_F(u, dX) \end{cases} \tag{3.14}$$

is locally Lipschitz continuous.

Proof. A direct calculation of $(F \circ u_1 - F \circ u_2)dX$ yields that we need to estimate a linear combination of the following distributions

$$
\begin{gathered}
T_{F\circ u_1 - F\circ u_2} dX, \quad T_{dX}(F\circ u_1 - F\circ u_2), \quad \Pi_F(u_1, dX) - \Pi_F(u_2, dX),\\
(F'\circ u_1)v_1 R(X, dX) - (F'\circ u_2)v_2 R(X, dX),\\
(F'\circ u_1)\Gamma(v_1, X, dX) - (F'\circ u_2)\Gamma(v_2, X, dX)\\
(F'\circ u_1)R(\mathbb{X}_T\mathcal{R}^{u_1}_{v_1}, dX) - (F'\circ u_2)R(\mathbb{X}_T\mathcal{R}^{u_2}_{v_2}, dX).
\end{gathered}
$$

Hence we have to estimate

$$
\begin{gathered}
(F'\circ u)(v_1 - v_2)R(X, dX) + (F'\circ u_1 - F'\circ u_2)v_2 R(X, dX)\\
(F'\circ u_1 - F'\circ u_2)\Gamma(v_1, X, dX) + (F'\circ u_2)\Gamma(v_1 - v_2, X, dX)\\
(F'\circ u_1 - F'\circ u_2)R(\mathbb{X}_T\mathcal{R}^{u_1}_{v_1}, dX) + (F'\circ u_2)R(\mathbb{X}_T\mathcal{R}^{u_2}_{v_1} - \mathbb{X}_T\mathcal{R}^{u_2}_{v_2}, dX).
\end{gathered}
$$

This is done standardly by applying the commutator estimates in Section 1.4.3, from which we arrive at the inequality

$$
\begin{aligned}
\|(F\circ u_1 - F\circ u_2)dX\|_{\mathcal{C}^{s-1}} \le C(s, X)\|F\|_{C^2_b} &\sum_{j=1,2}(\|u_j\|_{\mathcal{C}^s} + \|v_j\|_{\mathcal{C}^s} + \|\mathbb{X}_T\mathcal{R}^{u_j}_{F\circ u_j}\|)\\
&\times (1 + \|u_1 - u_2\|_{\mathcal{C}^s})(\|v_1 - v_2\|_{\mathcal{C}^s} + \|\mathbb{X}_T\mathcal{R}^{u_1}_{F\circ u_1} - \mathbb{X}_T\mathcal{R}^{u_2}_{F\circ u_2}\|_{\mathcal{C}^{2s}})
\end{aligned}
$$

for a positive constant $C(s, X)$. □

With this product operator at hand, it is now stright forward to see that if X has compact support, then the unique solution $u \in \mathcal{C}^s$ we constructed in Proposition 3.2.7 is also a paracontrolled distributions which solves the (RDE) (3.1) weakly.[4]

Theorem 3.2.9. *Let s be in $(1/3, 1)$ and $F \in C^3_b$. For every $\mathbb{X}_T \in \mathcal{X}^s_T$ the approximate solution constructed in Theorem 3.2.7 is the unique distribution in $\mathbb{X}\mathcal{D}^s_T$ which solves the (RDE) (3.1) driven by $\mathbb{X}_T$ weakly.*

Before concluding this section, we briefly state an alternate estimate of paracontrolled distributions, which will be made use of in Chapter Three. We note that in general paracontrooled distributions go beyond (RDE) and can be extended to treat many other equations. If s is an arbitrary real number of $\sigma > 0$, we say that $f \in \mathcal{C}^s$ is paracontrolled by $u \in \mathcal{C}^s$ if there exists $f' \in \mathcal{C}^\sigma$ such that

$$
\mathcal{R}_f \overset{\text{def}}{=} f - T_{f'}u
$$

is in $\mathcal{C}^{s+\sigma}$. We denote such a pair by (f, f').

Proposition 3.2.10. *Let (s, r) be in $(1/3, 1/2)$ and (u, v) in$\mathcal{C}^s \times \mathcal{C}^{s-1}$. Let (f, f') and (g, g') be paracontrolled by u and v respectively. Assume that $R(u, v) \in \mathcal{C}^{2s-1}$ can be constructed as the limit $\lim_{\epsilon\to 0} R(u^\epsilon, v^\epsilon)$ in $\mathcal{C}^{2s-1}$, where $(u^\epsilon)_{\epsilon>0}$ and $(v^\epsilon)_{\epsilon>0}$ are sequence*

[4] The proof of this theorem requires the author to extensively extend the length of this document, and therefore had to be omitted. See [4] for the detailed proof.

of smooth functions converging to u *and* v *respectively in* $\mathcal{C}^s$ *and* $\mathcal{C}^{s-1}$ *as* ϵ *goes to zero. Then the product* fg *is well-defined and satisfy the following estimate:*

$$\|fg - T_f g\|_{\mathcal{C}^{2s-1}} \leq C_{s,r}(\|f'\|_{\mathcal{C}^r}\|u\|_{\mathcal{C}^s} + \|\mathcal{R}_f\|_{\mathcal{C}^{s+r}})(\|g'\|_{\mathcal{C}^r}\|v\|_{\mathcal{C}^{s-1}} + \|\mathcal{R}_g\|_{\mathcal{C}^{s+r-1}}) + \|f'g'\|_{\mathcal{C}^r}\|R(u,v)\|_{\mathcal{C}^{2s-1}}.$$

Moreover, fg *depends locally Lipschitz continuously in the following sense: Let* $(\tilde{u},\tilde{v}) \in \mathcal{C}^s \times \mathcal{C}^{s-1}$ *with* $R(\tilde{u},\tilde{v}) \in \mathcal{C}^{2s-1}$ *such that* $(\tilde{f},\tilde{f}')$ *and* $(\tilde{g},\tilde{g}')$ *are paracontrolled by* $\tilde{u}$ *and* $\tilde{v}$ *respectively. Suppose there exists an upper bound* $M > 0$ *of all norms under considerations, then we have*

$$\|(fg - T_f g) - (\tilde{f}\tilde{g} - T_{\tilde{f}}\tilde{g})\|_{\mathcal{C}^{2s-1}} \leq C_{s,r}(1+M^3)\Big(\|f' - \tilde{f}'\|_{\mathcal{C}^r} + \|g' - \tilde{g}'\|_{\mathcal{C}^r} + \|u - \tilde{u}\|_{\mathcal{C}^s} + \|v - \tilde{v}\|_{\mathcal{C}^{s-1}} + \|\mathcal{R}_f - \mathcal{R}_{\tilde{f}}\|_{\mathcal{C}^{s+r}} + \|\mathcal{R}_g - \mathcal{R}_{\tilde{g}}\|_{\mathcal{C}^{s+r-1}} + \|R(u,v) - R(\tilde{u},\tilde{v})\|_{\mathcal{C}^{2s-1}}\Big).$$

In particular, we can replace M^3 *by* M^2 *if* $f = \tilde{f}' = 0$ *or* $g = \tilde{g}' = 0$.

The proof will be omitted since it is similar to any other Lipschitz continuity results that we have proven in this Chapter and thus contains no real difficulty.

3.3 References and Remarks

Historically, the theory of rough differential equations were first considered by Terry Lyons and his theory of rough paths. See the introductory discussion or [5] for a more complete treatment. Young's integral was of course first considered by L.C.Young, which subsequently motivated Lyons for his construction of integration against geometric rough paths. Our construction of the rough integral from this point of view is original but was motivated by [3], there the author reproved Young's theory using an alternate characterisation of Besov spaces, namely Haar basis, to give an equivalent formulation of the results we proved here. Most of our analysis on the (RDE) can be found in [2] but we in particular also followed the approach in [4]. The latter was more suitable for our chronological discussion on the subject. The theory centralising around paracontrolled distributions can be considered as a collection of estimate which can be adapted accordingly to be useful in many places. Due to Young's theorem, our definition in 3.1.5 makes an intuitive example of this but the situation could certainly be more general, as illuminated in Proposition 3.2.10. In Chapter Three we will witness another application of paracontrolled distributions, although there how the structure is involved is less clear.

We also didn't have to stop where we did. By interpreting the rough signal as the Brownian motion B, we could now directly calculate smooth regularisations $(B^\epsilon, R(B^\epsilon, dB^\epsilon))_{\epsilon>0}$ which converges to (B, dB) almost surely in $\mathcal{C}^s \times \mathcal{C}^{s-1}$. Such a construction will be entirely probabilistic. By theorem 3.2.1, we then arrive at the existence of a unique solution to the stochastic ordinary differential equation:

$$\begin{cases} \dfrac{d}{dt}u(t,\omega) = F(u(t,\omega))dB(\omega), \\ u(0,\omega) = u_0(\omega), \quad \omega \in \Omega \end{cases}$$

for some suitable probability space (Ω, P). See also [4] for a treatment of this result. We could also solve any other equations of this form with the stochastic noise B being replaced by processes which are in $\mathcal{C}^s$ almost surely for $s \in (1/3, 1/2)$. Notice that in classical stochastic analysis, such an equation could be solved probabilistically using Ito's integral, with Theorem 3.2.1 being an optional deterministic approach. However, in the more difficult situations of partial differential equations, there exists equations for which stochastic analysis tend to fail, in which case such an alternative would be desirable.

Chapter 4

The Kardar-Parisi-Zhang Equation

In this chapter we will begin our mathematical study of the most important example of singular stochastic partial differential equations: the one-dimensional *Kardar-Parisi-Zhang Equation* (KPZ). Deterministically, the equation is formally stated as

$$\partial_t h(x,t) - \nu \partial_x^2 h(x,t) = \frac{\lambda}{2}(\partial_x h)^2(x,t) + \sqrt{D} Z(x,t), \quad h_{|t=0} = h_0, \tag{4.1}$$

where Z is the rough analogue of the space-time white noise, h is a continuous function is both space and time, and $(\nu, \lambda, D) \in \mathbb{R}^2 \times (0,\infty)$ are parameters. If we consider the equation with periodic boundary conditions, then one could show stochastically that Z is in $C(\mathcal{C}^{-1/2-})$. However, on the real line the spatial variable of the white noise grows too fast to achieve this level of regularity. Thus we assume $h \in C((0,\infty); C(\mathbb{T};\mathbb{R}))$. The Hölder-Besov spaces in this chapter will also be adjusted without further comment. Moreover, since we will only be dealing with estimates, without loss of generality we may assume that ν, D are one and that $\lambda = 2$. As in the case of (RDE), the low regularity of Z stipulates that the product $(\partial_x h)^2$ is singular. Therefore the (KPZ) defines a singular stochastic partial differential equation. We will see how similar techniques which were utilised in the previous section extend to the (KPZ). In particular, we construct a suitable space of paracontrolled distributions in which a well-defined solution can be found. It is worth noting, however, that since the noise term in the (KPZ) is additive, we do not necessarily expect the result to be weaker than the case of (RDE), where the noise is multiplicative. We will show that this is indeed the case.

4.1 The Rough Burger's Equation

Instead of directly tackling the (KPZ), it is more convenient to study another closely related partial differential equation. Let h be the solution to the one-dimensional (KPZ) and set $u \overset{\text{def}}{=} \partial_x$h. Then it is obvious that

$$\mathcal{L}u = \partial_x(\partial_x h)^2 + \partial_x Z = \partial_x u^2 + \partial_x Z,$$

with intiial condition $u_{|t=0} = \partial_x h_{|t=0}$. Hence formally, the (KPZ) is pathwise equivalent to the *Rough Burger's Equation* (RBE):

$$\begin{cases} \mathcal{L}u = \partial_x u^2 + \partial_x Z \\ u_{|t=0} = u_0. \end{cases} \tag{4.2}$$

In this section, we will be concerned with the study of well-posedness result for the (RBE).

4.1.1 Structure of the Solution

As in the case of (RDE) we would like to obtain a certain structure of the solution to the (RBE) and define the corresponding space of paracontrolled distributions, such that if the stochastic component could be constructed via smooth approximation, then the singularity in the equation could be properly interpreted. Observe from (4.2) that the equation is ill-posed because there is no obvious way to define the non-linear term $\partial_x u^2$. Moreover, dropping such a term yields a (H) driven by an external force $\partial_x Z$, with the corresponding solution well-known and is given by

$$Y(t,x) \stackrel{\text{def}}{=} (V_t \circ \partial_x Z)(t,x) = \int_0^t e^{(t-t')\Delta}\partial_x Z(t',x)dt'.$$

If we can separate the solution u additively into Y plus another term that solves the non-linear component of the (RBE). Then we expect

$$u = Y + R^{(1)}u, \quad \text{with} \quad \mathcal{L}R^{(1)}u = \partial_x Y^2 + 2\partial_x(YR^{(1)}u) + \partial_x(R^{(1)}u)^2.$$

We can imagine to further linearise by removing more undesirable terms under the action of $\mathcal{L}$. However, to simplify notation, we would like to specify more compact notation for the terms which will appear in our increasingly complicated expansions. More precisely, we would like to introduce the bilinear operator:

$$\mathcal{B} : \mathcal{D}' \times \mathcal{D}' \to \mathcal{D}' : (u,v) \longmapsto V_t \circ \partial_x(uv) = \int_0^t e^{(t-t')\Delta}\partial_x(uv)(t',x)dt',$$

as well as the notations

$$Y_{\vee} \stackrel{\text{def}}{=} \mathcal{B}(Y,Y),\ Y_{\vee_{\vee}} \stackrel{\text{def}}{=} \mathcal{B}(Y,Y_{\vee}),\ Y_{\vee_{\vee_{\vee}}} \stackrel{\text{def}}{=} \mathcal{B}(Y,Y_{\vee_{\vee}}),\ Y_{\vee\!\vee} \stackrel{\text{def}}{=} \mathcal{B}(Y_{\vee},Y_{\vee}).$$

Notice that even the relatively simple term $Y_{\vee}$ cannot be easily defined in any Besov space, since we would like to interpret the product Y^2 using Bony's decomposition, for which the regularity requirement of $R(Y,Y)$ would be strictly violated. Actually , in order to make sense of the (RBE) one must construct the regularities

$$(Y, Y_{\vee}, Y_{\vee_{\vee}}, Y_{\vee_{\vee_{\vee}}}, Y_{\vee\!\vee}) \in C(\mathcal{C}^{-1/2-}) \times C(\mathcal{C}^{0-}) \times C(\mathcal{C}^{1/2-}) \times C(\mathcal{C}^{1/2-}) \times C(\mathcal{C}^{1-}).$$

Suppose for the moment that this could be done, then in this language, we can easily express further expansions of u:

$$\begin{aligned} u &= Y + Y_{\vee} + R^{(2)}u, \text{ with} \\ R^{(2)}u &= 2Y_{\vee_{\vee}} + Y_{\vee\!\vee} + 2\mathcal{B}(Y,\mathcal{R}^{(2)}u) + 2\mathcal{B}(Y_{\vee},\mathcal{R}^{(2)}u) + \mathcal{B}(\mathcal{R}^{(2)}u,\mathcal{R}^{(2)}u) \end{aligned}$$

and so on, with the aim of controlling the expansion under the regularities given above. To achieve this for the case of $R^{(2)}u$, every term in the expansion must also be well-defined in some suitable Besov spaces. However, it is not hard to see that the requirement would be for $R^{(2)}u$ to be in at most $C(\mathcal{C}^{1/2-})$, in which case we have $\mathcal{B}(R^{(2)}u, R^{(2)}u) \in C(\mathcal{C}^{3/2-})$. But with this condition, the term $\mathcal{B}(Y, R^{(2)}u)$ cannot be defined. This implies further expansion around $R^{(2)}u$ is necessary. A straightforward approach would be to set

$$\begin{aligned} R^{(2)}u &= 2Y_{\vee_{\vee}} + R^{(3)}(u), \text{ with} \\ R^{(3)}u &= 4Y_{\vee_{\vee_{\vee}}} + Y_{\vee\!\vee} + 2\mathcal{B}(Y,R^{(3)}u) + 2\mathcal{B}(Y_{\vee},R^{(2)}u) + \mathcal{B}(R^{(2)}u,R^{(2)}u). \end{aligned}$$

If we are willing to assume $(R^{(2)}u,\ R^{(3)}u) \in C(\mathcal{C}^{1/2-}) \times C(\mathcal{C}^{1/2-})$, then we can conclude that

$$(Y_{\substack{\vee\vee\\\vee}},\ \mathcal{B}(Y_{\vee}, R^{(2)}u),\ \mathcal{B}(R^{(2)}u, R^{(2)}u)) \in C(\mathcal{C}^{1-}) \times C(\mathcal{C}^{1-}) \times C(\mathcal{C}^{3/2-}).$$

Hence in this case, we are able to deduce

$$R^{(3)}u - 4Y_{\substack{\vee\\\vee\\\vee}} - 2\mathcal{B}(Y, R^{(3)}u) \in C(\mathcal{C}^{1-}).$$

Inspired by such an expansion, we propose the following paracontrolled structure:

$$\begin{cases} u = Y + Y_{\vee} + 2Y_{\substack{\vee\\\vee}} + \Theta_u, \\ \Theta_u \overset{\text{def}}{=} \tilde{T}_{2\Theta u + 4Y_{\substack{\vee\\\vee}}}\Theta + \mathcal{R}u,\ \mathcal{L}\Theta = \partial_x Y,\ \mathcal{R}u \in C(\mathcal{C}^{1-}). \end{cases} \tag{4.3}$$

To see this is the correct structure for a solution to the (RBE), notice that the non-linear component takes the form

$$\begin{aligned} \partial_x u^2 &= \partial_x(Y^2 + 2YY_{\vee} + Y_{\vee}^2 + 4YY_{\substack{\vee\\\vee}}) \\ &\quad + 2\partial_x(\Theta_u Y) + 2\partial_x(Y_{\vee}(\Theta_u + 2Y_{\substack{\vee\\\vee}})) + \partial_x((\Theta_u + 2Y_{\substack{\vee\\\vee}})(\Theta_u + 2Y_{\substack{\vee\\\vee}})). \end{aligned}$$

Because

$$\|\tilde{T}_{2\Theta_u + 4Y_{\substack{\vee\\\vee}}}\Theta\|_{C_T(\mathcal{C}^{1/2-\epsilon})} \le C\|2\Theta + 4Y_{\substack{\vee\\\vee}}\|_{C_T(\mathcal{C}^{(1-\epsilon)/2})}\|\Theta\|_{C_T(\mathcal{C}^{(1-\epsilon)/2})}, \quad \epsilon > 0,$$

the paraproduct $\tilde{T}_{2\Theta_u + 4Y_{\substack{\vee\\\vee}}}\Theta$ is in $C(\mathcal{C}^{1/2-})$. By our hypothesis on the regularity of $\mathcal{R}u$, we readily obtain $\Theta_u \in C(\mathcal{C}^{1/2-})$. This leads to the observation that

$$(Y_{\vee}(\Theta_u + Y_{\substack{\vee\\\vee}}), (\Theta_u + Y_{\substack{\vee\\\vee}})^2) \in C(\mathcal{C}^{0-}) \times C(\mathcal{C}^{1/2-})$$

and so $2\partial_x(Y_{\vee}(\Theta_u + Y_{\substack{\vee\\\vee}})) + \partial_x((\Theta_u + Y_{\substack{\vee\\\vee}})^2) \in C(\mathcal{C}^{-1-})$ exists. Since

$$\partial_x(Y^2 + 2YY_{\vee} + Y_{\vee}^2 + 4YY_{\substack{\vee\\\vee}}) = \mathcal{L}(Y_{\vee} + 2Y_{\substack{\vee\\\vee}} + Y_{\substack{\vee\vee\\\vee}} + 4Y_{\substack{\vee\\\vee\\\vee}}),$$

the only term which remains to be defined is $\partial_x(\Theta_u Y)$, which we would like to interpret via Bony's decomposition provided that $R(\Theta_u, Y)$ exists. In particular, if we assume the existence of a family of smooth functions $(\Theta_u^\epsilon, Y^\epsilon)$ such that $\lim_{\epsilon\to 0} R(\Theta_u^\epsilon, Y^\epsilon) = R(\Theta_u, Y)$ in $C(\mathcal{C}^{0-})$, then under this assumption, we can easily deduce from Proposition 3.2.10 that

$$\begin{aligned} &(\partial_x(\Theta_u Y) - T_{\partial_x Y}\Theta_u - T_Y\partial_x\Theta_u, T_Y\partial_x\Theta_u) \in C(\mathcal{C}^{-1-}) \times C(\mathcal{C}^{-1/2-}) \\ &\qquad\qquad\qquad\qquad \Longrightarrow \partial_x(\Theta_u Y) - T_{\partial_x Y}\Theta_u \in C(\mathcal{C}^{-1-}). \end{aligned}$$

Therefore in this case, the term $\partial_x u^2$ can be made sense of provided u satisfies the appropriate paracontrolled structure. Moreover, suppose that there exists another structure

$$\begin{cases} u = Y + Y_{\vee} + 2Y_{\substack{\vee\\\vee}} + \tilde{\Theta}_u, \\ \tilde{\Theta}_u \overset{\text{def}}{=} \tilde{T}_v\tilde{\Theta} + \tilde{\mathcal{R}}u,\ (v, \tilde{\Theta}, \tilde{\mathcal{R}}u) \in \mathcal{L}^{1/2-} \times C(\mathcal{C}^{1/2-}) \times C(\mathcal{C}^{1-}). \end{cases}$$

Then we can derive an equation for the corresponding remainder $\tilde{\mathcal{R}}u$,

$$\mathcal{L}\tilde{\mathcal{R}}u = \partial_x u^2 - \partial_x Y^2 - 2\partial_x(YY_{\vee}) - \mathcal{L}\tilde{T}_v\tilde{\Theta}.$$

Notice that we can apply Lemma 2.5.1 which yields the commutation estimates

$$(\mathcal{L}\tilde{T}_v\tilde{\Theta} - \tilde{T}_v\mathcal{L}\tilde{\Theta}, \tilde{T}_{\Theta_u}\partial_x Y - T_{\Theta_u}\partial_x Y) \in C(\mathcal{C}^{-1-}) \times C(\mathcal{C}^{-1-}). \tag{4.4}$$

Hence, combining with our previous expansion for $\partial_x u^2$, we have

$$\mathcal{L}\tilde{\mathcal{R}}u - 4\partial_x(YY_{\vee_{\vee}}) - 2T_{\tilde{\Theta}_u}\partial_x Y + \tilde{T}_v\mathcal{L}\tilde{\Theta} = (\tilde{T}_v\mathcal{L}\Theta - \mathcal{L}\tilde{T}_{2\Theta_u+4Y_{\vee_{\vee}}}\tilde{\Theta}) + \partial_x Y_{\vee}^2$$
$$+ 2\partial_x(Y_{\vee}(\tilde{\Theta}_u + 2Y_{\vee_{\vee}})) + \partial_x((\tilde{\Theta}_u + 2Y_{\vee_{\vee}})^2) + 2T_{\partial_x Y}\tilde{\Theta}_u + 2R(\partial_x Y, \tilde{\Theta}_u) + 2Y\partial_x\tilde{\Theta}_u,$$

with the right hand side being in $C(\mathcal{C}^{-1-})$. If we can construct $R(Y, Y_{\vee})$ such that it exists in $C(\mathcal{C}^{0-})$, then in particular we can expree

$$C(\mathcal{C}^{-1-}) \ni \partial_x(YY_{\vee_{\vee}}) = T_{Y_{\vee_{\vee}}}\partial_x Y + \partial_x T_Y Y_{\vee_{\vee}} + T_{\partial_x Y_{\vee_{\vee}}} Y + \partial_x R(Y, Y_{\vee_{\vee}}).$$

Thus (4.4) shows that we have

$$\mathcal{L}\tilde{\mathcal{R}}u - \tilde{T}_{2\Theta_u+4Y_{\vee_{\vee}}}\partial_x Y + \tilde{T}_v\mathcal{L}\tilde{\Theta} \in C(\mathcal{C}^{-1-}).$$

In order to preserve the regularity at the level of $C(\mathcal{C}^{-1-})$ we thus need to choose $(v, \mathcal{L}\tilde{\Theta}) = (2\Theta_u + 4Y_{\vee_{\vee}}, \partial_x Y)$. This shows that (4.3) is the correct structure for a solution to the (RBE).

In summary, for any distribution u satisfying the paracontrolled structure (4.3) with the associated regularity conditions, we can make sense of the non-linearity of the (RBE) provided we can define

$$(Y, Y_{\vee}, Y_{\vee_{\vee}}, Y_{\vee_{\vee_{\vee}}}, R(Y, Y_{\vee}), R(\Theta, Y))$$

in the appropriate Besov spaces

$$C(\mathcal{C}^{-1/2-}) \times C(\mathcal{C}^{0-}) \times C(\mathcal{C}^{1/2-}) \times C(\mathcal{C}^{1/2-}) \times C(\mathcal{C}^{0-}) \times C(\mathcal{C}^{0-}).$$

The term $R(\Theta, Y)$ must be defined in the sense of smooth approximation. Notice that the condition on $R(Y, Y_{\vee})$ already ensures the regularity of $\mathcal{B}(Y, Y_{\vee})$ in $C(\mathcal{C}^{1/2-})$ provided all the other data are sufficiently regular. Hence the above requirements are consistent with the earlier arguments. In particular, by interpreting the (RBE) via expansion of $\partial_x u^2$, one sees that u completely satisfies and therefore solves the (RBE).

This leads to the following abstract definition[1].

Definition 4.1.1. Let s be in $(1/3, 1/2)$. The space of (RBE) enhanceable elements $\mathcal{Y}^s$ consists of all those distributions $\mathbb{Y}$ in the closure for the image of the map

$$\mathcal{L}C(\mathbb{R}; \mathcal{D}(\mathbb{T})) \to C(\mathcal{C}^{s-1}) \times C(\mathcal{C}^{2s-1}) \times \mathcal{L}^s \times \mathcal{L}^{2s} \times \mathcal{L}^{2s} \times C(\mathcal{C}^{2s-1})$$

defined by

$$Z \longmapsto (Y, Y_{\vee}, Y_{\vee_{\vee}}, \mathring{Y}_{\vee_{\vee_{\vee}}}, Y_{\vee\vee}, R(\Theta, Y))$$

with the elements satisfying

$$\mathcal{L}Y = \partial_x Z, \quad \mathcal{L}Y_{\vee} = \partial_x Y^2, \quad \mathcal{L}Y_{\vee_{\vee}} = \partial_x(YY_{\vee}),$$

$$\mathcal{L}\mathring{Y}_{\vee_{\vee_{\vee}}} = \partial_x R(Y_{\vee_{\vee}} Y), \quad \mathcal{L}Y_{\vee\vee} = \partial_x(Y_{\vee}Y_{\vee}), \quad \mathcal{L}\Theta = \partial_x Y,$$

[1] The notation introduced in this definition would be somewhat tedious and readers should have enough reasons to wonder why would anyone even make use of them. See the reference and remarks section for an immediate explanation.

and initial conditions

$$(Y_{\vee}, Y_{\vee_{\vee}}, \mathring{Y}_{\vee_{\vee_{\vee}}}, Y_{\vee\vee}, \Theta)_{|t=0} = 0 \quad \text{and} \quad Y_{|t=0} = \int_{-\infty}^{0} e^{-t\Delta}(\partial_x W)dt.$$

pointwise in $\mathcal{L}C(\mathbb{R}; \mathcal{D}(\mathbb{T}))$. The initial condition of Y is also denoted by Y_0. For every $T > 0$, the normed space $(\mathcal{Y}^s_T, \|\cdot\|_{\mathcal{Y}^s_T})$ is

$$\mathcal{Y}^s_T \stackrel{\text{def}}{=} \mathcal{Y}^s{}_{|[0,T]} \quad \mathbb{Y}_T \stackrel{\text{def}}{=} \mathbb{Y}_{|[0,T]} \text{ for every } \mathbb{Y} \in \mathcal{Y}^s, \quad \text{and}$$

$$\|\mathbb{Y}_T\|_{\mathcal{Y}^s_T} \stackrel{\text{def}}{=} \|Y\|_{C_T(\mathcal{C}^{s-1})} + \|Y_{\vee}\|_{C_T(\mathcal{C}^{2s-1})} + \|Y_{\vee_{\vee}}\|_{\mathcal{L}^s_T} + \|\mathring{Y}_{\vee_{\vee_{\vee}}}\|_{\mathcal{L}^{2s}_T} + \|Y_{\vee\vee}\|_{\mathcal{L}^{2s}_T} + \|R(\Theta, Y)\|_{C_T(\mathcal{C}^{2s-1})}.$$

As in the case of (RDE), for each (RBE) enhanceable element we can also consider a space of paracontrolled distributions.

Definition 4.1.2. Let $s \in (1/3, 1/2)$ and $\mathbb{Y}$ be in $\mathcal{Y}^s$ and suppose that $r > 1 - 2s$. A pair of distributions $(u, v) \in C(\mathcal{C}^{s-1}) \times \mathcal{L}^r$ is *paracontrolled by* $\mathbb{Y}$ if there exists $\mathbb{Y}\mathcal{R}^u_v \in \mathcal{L}^{s+r}$ such that

$$u = Y + Y_{\vee} + 2Y_{\vee_{\vee}} + \tilde{T}_v \Theta + \mathbb{Y}\mathcal{R}^u_v.$$

We say that v is the derivative of u paracontrolled by $\mathbb{Y}$. The space $\mathbb{Y}\mathcal{D}^r$ consists of all such distributions

$$\mathbb{Y}\mathcal{D}^r \stackrel{\text{def}}{=} \{(u, v) \in C(\mathcal{C}^{s-1}) \times \mathcal{L}^r \mid \mathbb{Y}\mathcal{R}^u_v \stackrel{\text{def}}{=} u - Y - Y_{\vee} - 2Y_{\vee_{\vee}} - \tilde{T}_v \Theta \in \mathcal{L}^{s+r}\}.$$

For every $T > 0$, the normed space $(\mathbb{Y}\mathcal{D}^r_T, \|\cdot\|_{\mathbb{Y}\mathcal{D}^r_T})$ is

$$\mathbb{Y}\mathcal{D}^r_T \stackrel{\text{def}}{=} \mathbb{Y}\mathcal{D}^r{}_{|[0,T]} \quad \text{and}$$

$$\|(u, v)\|_{\mathbb{Y}\mathcal{D}^r_T} \stackrel{\text{def}}{=} \|v\|_{\mathcal{L}^r_T} + \|\mathbb{Y}\mathcal{R}^u_v\|_{\mathcal{L}^{s+r}_T}.$$

It is clear that $(\mathbb{Y}\mathcal{D}^r_T, \|\cdot\|_{\mathbb{Y}\mathcal{D}^r_T})$ is complete. In the setting of the (RBE) we will often implicitly choose the paracontrolled structure $(u, 2\Theta_u + 4Y_{\vee_{\vee}})$ to be compatible with (4.3), with regularity $s \in (1/3, 1/2)$ in place of $1/2-$, such that u solves the corresponding (RBE) driven by $\mathbb{Y}$. Every such u is a *paracontrolled solution* to the (RBE). Under this context, we would also make use of the notation

$$\|u\|_{\mathbb{Y}\mathcal{D}^r_T} \stackrel{\text{def}}{=} \|2\Theta_u + 4Y_{\vee_{\vee}}\|_{\mathcal{L}^r_T} + \|\mathbb{Y}\mathcal{R}^u_{2\Theta_u + 4Y_{\vee_{\vee}}}\|_{\mathcal{L}^{s+r}_T}, \quad T > 0.$$

Clearly, there is no ambiguity. Our analysis of the (RBE) will be centralised around these two spaces hereafter.

4.1.2 Well-posedness of the (*RBE*) with $\mathcal{C}^{2s}$ Data

In this subsection we will be devoted to establishing the well-posedness of the (RBE) driven by signals in $\mathcal{Y}^s$, which will be shown to exists up to blow-up time.

Theorem 4.1.3. *Let $(\mathbb{Y}, u_0)$ be in $\mathcal{Y}^s \times \mathcal{C}^{2s}$. Then there exists a unique paracontrolled solution $(u, v) \in \mathbb{Y}\mathcal{D}^s$ to the (RBE) with initial condition $Y_0 + u_0$ for all $T < T^*$, where*

$$T^* \overset{def}{=} \inf_{\substack{T \in \mathbb{R}^+ \\ \|(u,v)\|_{\mathbb{Y}\mathcal{D}^s_T} = \infty}} T,$$

in the sense that (u, v) is the unique fixed point to the mapping

$$\begin{cases} \mathbb{Y}\mathcal{D}^s \to \mathbb{Y}\mathcal{D}^s \\ (u,v) \mapsto (e^{t\Delta}u_0 + V_t(\partial_x u^2) + Y, 2\Theta_u + 4Y_{\vee_\vee}), \ \ \Theta_u \overset{def}{=} \tilde{T}_v \Theta + \mathbb{Y}\mathcal{R}^u_v. \end{cases}$$

Although the theorem will be proved using relatively clean arguments, still some technical estimates cannot be avoided. These will be summarized in the following lemma.

Lemma 4.1.4. *Suppose that $r > 1 - 2s$, then we have*

$$\begin{aligned} \|\mathcal{L}\mathbb{Y}\mathcal{R}^u_{2\Theta_u + 4Y_{\vee_\vee}} - \mathcal{L}Y_{\vee\vee} - 4\mathcal{L}\mathring{Y}_{\vee_{\vee_\vee}}\|_{C_T(\mathcal{C}^{2s-2})} + \|2\Theta_u + 4Y_{\vee_\vee}\|_{\mathcal{L}^s_T} \\ \leq C(1 + M_T^2)(1 + \|u\|^2_{\mathbb{Y}\mathcal{D}^r_T}). \end{aligned}$$

Additionally, suppose that $(\tilde{u}, 2\tilde{\Theta}_{\tilde{u}} + 4\tilde{Y}_{\vee_\vee})$ is another distribution paracontrolled by $\tilde{\mathbb{Y}} = (\tilde{Y}, \tilde{Y}_{\vee_\vee}, \mathring{\tilde{Y}}_{\vee_{\vee_\vee}}, \tilde{Y}_{\vee\vee}, \tilde{\Theta}) \in \mathcal{Y}^s$ such that

$$\max\{\|(u, 2\Theta_u + 4Y_{\vee_\vee})\|_{\mathbb{Y}\mathcal{D}^s_T}, \|(\tilde{u}, 2\tilde{\Theta}_{\tilde{u}} + 4\tilde{Y}_{\vee_\vee})\|_{\tilde{\mathbb{Y}}\mathcal{D}^s_T}\} \leq M_T.$$

Then we have

$$\begin{aligned} \|(\mathcal{L}\mathbb{Y}\mathcal{R}^u_{2\Theta_u + 4Y_{\vee_\vee}} - \mathcal{L}Y_{\vee\vee} - 4\mathcal{L}\mathring{Y}_{\vee_{\vee_\vee}}) - (\mathcal{L}\tilde{\mathbb{Y}}\mathcal{R}^{\tilde{u}}_{2\tilde{\Theta} + 4Y_{\vee_\vee}} - \mathcal{L}\tilde{Y}_{\vee\vee} - 4\mathcal{L}\mathring{\tilde{Y}}_{\vee_{\vee_\vee}})\|_{C_T(\mathcal{C}^{2s-2})} \\ + \|2(\Theta_u + \tilde{\Theta}_{\tilde{u}}) + 4(\mathring{Y}_{\vee_{\vee_\vee}} - \mathring{\tilde{Y}}_{\vee_{\vee_\vee}})\|_{\mathcal{L}^s_T} \\ \leq M_T^2(\|\mathbb{Y} - \tilde{\mathbb{Y}}\|_{\mathcal{Y}^s_T} + \|2(\Theta_u + \tilde{\Theta}_{\tilde{u}}) + 4(\mathring{Y}_{\vee_{\vee_\vee}} - \mathring{\tilde{Y}}_{\vee_{\vee_\vee}})\|_{\mathcal{L}^r_T} + \|\mathbb{Y}\mathcal{R}^u_{2\Theta_u + 4Y_{\vee_\vee}} - \tilde{\mathbb{Y}}\mathcal{R}^{\tilde{u}}_{2\tilde{\Theta}_{\tilde{u}} + 4\tilde{Y}_{\vee_\vee}}\|_{\mathcal{L}^{2r}}). \end{aligned}$$

Proof. By applying the paracontrolled structure, we have

$$\begin{aligned} \mathcal{L}\mathbb{Y}\mathcal{R}^u_{2\Theta_u + 4Y_{\vee_\vee}} &= \mathcal{L}u - \mathcal{L}Y - \mathcal{L}Y_{\vee} - 2\mathcal{L}Y_{\vee_\vee} - \mathcal{L}\tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\Theta \\ &= \partial_x u^2 - \partial_x Y^2 - 2\partial_x(YY_{\vee}) - \mathcal{L}\tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\Theta \\ &= \partial_x Y_{\vee}^2 + 2\partial_x(\Theta_u Y) + 4\partial_x(YY_{\vee_\vee}) + 2\partial_x(\Theta_u Y_{\vee}) + 4\partial_x(Y_{\vee}Y_{\vee_\vee}) \\ &\quad + \partial_x \Theta_u^2 + 4\partial_x(\Theta_u Y_{\vee_\vee}) + 4\partial_x Y_{\vee_\vee}^2 - \mathcal{L}\tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\Theta. \end{aligned}$$

By considering the terms $2\partial_x T_{\Theta_u} Y$ and $\tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\partial_x Y$, and because

$$T_{2\Theta_u + 4Y_{\vee_\vee}}\partial_x Y - 2T_{\Theta_u}\partial_x Y - 4T_{Y_{\vee_\vee}}\partial_x Y = 0,$$

we have

$$\begin{aligned} \mathcal{L}\mathbb{Y}\mathcal{R}^u_{2\Theta_u + 4Y_{\vee_\vee}} &= \mathcal{L}Y_{\vee\vee} + 2\partial_x(\Theta_u Y - T_{\Theta_u}Y) + 2(\partial_x T_{\Theta_u}Y - T_{\Theta_u}\partial_x Y) + 4\mathcal{L}\mathring{Y}_{\vee_{\vee_\vee}} \\ &\quad + 4\partial_x T_Y Y_{\vee_\vee} + 4(\partial_x T_{Y_{\vee_\vee}} Y - T_{Y_{\vee_\vee}}\partial_x Y) + (T_{2\Theta_u + 4Y_{\vee_\vee}}\partial_x Y - \tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\partial_x Y) \\ &\quad + \partial_x(4Y_{\vee}Y_{\vee_\vee} + 2\Theta_u Y_{\vee}) + \partial_x(2Y_{\vee_\vee} + \Theta_u)^2 - (\mathcal{L}\tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\Theta - \tilde{T}_{2\Theta_u + 4Y_{\vee_\vee}}\mathcal{L}\Theta). \end{aligned}$$

By the estimates

$$\|T_Y \Theta_u\|_{C_T(\mathcal{C}^{2s-1})} \leq C\|Y\|_{C_T(\mathcal{C}^{s-2})}\|\Theta_u\|_{C_T(\mathcal{C}^s)}$$
$$\|\partial_x T_{\Theta_u} Y - T_{\Theta_u}\partial_x Y\|_{C_T(\mathcal{C}^{2s-2})} \leq C\|\Theta_u\|_{C_T(\mathcal{C}^s)}\|Y\|_{C_T(\mathcal{C}^{s-1})}$$
$$\|\partial_x T_{Y_{\curlyvee}} Y - T_{Y_{\curlyvee}}\partial_x Y\|_{C_T(\mathcal{C}^{2s-2})} \leq C\|Y_{\curlyvee}\|_{C_T(\mathcal{C}^s)}\|Y\|_{C_T(\mathcal{C}^{s-1})}$$
$$\|\partial_x(4Y_{\vee}Y_{\curlyvee} + 2\Theta_u Y_{\vee})\|_{C_T(\mathcal{C}^{2s-2})} \leq C\|Y_{\vee}\|_{C_T(\mathcal{C}^{2s-1})}\|Y_{\curlyvee}\|_{\mathcal{L}^s_T} + C'\|\Theta_u\|_{C_T(\mathcal{C}^s)}\|Y_{\vee}\|_{C_T(\mathcal{C}^{2s-1})}$$
$$\|\partial_x(2Y_{\curlyvee} + \Theta_u)^2\|_{C_T(\mathcal{C}^{2s-2})} \leq C\|2\Theta_u + 4Y_{\curlyvee}\|^2_{C_T(L^\infty)}$$
$$\|T_{2\Theta_u+4Y_{\curlyvee}}\partial_x Y - \tilde{T}_{2\Theta_u+4Y_{\curlyvee}}\partial_x Y\|_{C_T(\mathcal{C}^{2s-2})} \leq \|2\Theta_u + 4Y_{\curlyvee}\|_{\mathcal{L}^s_T}\|Y\|_{C_T(\mathcal{C}^{s-1})}$$
$$\|\mathcal{L}\tilde{T}_{2\Theta_u+4Y_{\curlyvee}}\Theta - \tilde{T}_{2\Theta_x+4Y_{\curlyvee}}\mathcal{L}\Theta\|_{C_T(\mathcal{C}^{2s-2})} \leq \|2\Theta_u + 4Y_{\curlyvee}\|_{\mathcal{L}^s_T}\|\Theta\|_{\mathcal{L}^s_T}$$

we get

$$\begin{aligned}
&\|\mathcal{L}\mathbb{Y}\mathcal{R}^u_{2\Theta_u+4Y_{\curlyvee}} - \mathcal{L}Y_{\vee\vee} - 4\mathcal{L}\mathring{Y}_{\curlyvee}\|_{C_T(\mathcal{C}^{2s-2})}\\
&\quad\leq C\big(\|\mathbb{Y}\|_{\mathcal{Y}^s_T}(\|\Theta_u\|_{C_T(\mathcal{C}^s)} + \|2\Theta_u + 4Y_{\curlyvee}\|^2_{C_T(L^\infty)})\\
&\quad+ (\|\mathbb{Y}\|_{\mathcal{Y}^s_T} + \|\mathbb{Y}\|^2_{\mathcal{Y}^s_T})(1 + \|\mathbb{Y}\mathcal{R}^u_{2\Theta_u+4Y_{\curlyvee}}\|_{C_T(\mathcal{C}(\mathcal{C}^{s+r}))} + \|2\Theta_u + 4Y_{\curlyvee}\|_{C_T(\mathcal{C}^r)})\big),
\end{aligned}$$

where we have used that $\|\Theta\|_{\mathcal{L}^s_T} \leq C\|\mathbb{Y}\|_{\mathcal{Y}^s_T}$ by the Schauder estimate. Now,

$$\begin{aligned}
\|\Theta_u\|_{C_T(\mathcal{C}^s)} &\leq C\big(\|2\Theta_u + 4Y_{\curlyvee}\|_{\mathcal{L}^r_T}(\|\Theta\|_{C_T(\mathcal{C}^s)} + \|\mathcal{L}\Theta\|_{C_T(\mathcal{C}^{s-2})}) + \|\mathbb{Y}\mathcal{R}^u_{2\Theta_u+4Y_{\curlyvee}}\|_{C_T(\mathcal{C}^s)}\big)\\
&\leq 2C\big(\|2\Theta_u + 4Y_{\curlyvee}\|_{\mathcal{L}^r_T}\|\mathbb{Y}\|_{\mathcal{Y}^s_T} + \|\mathbb{Y}\mathcal{R}^u_{2\Theta_u+4Y_{\curlyvee}}\|_{C_T(\mathcal{C}^s)}\big).
\end{aligned}$$

Furthermore, by combining an application of Corollary 2.5.7, the paracontrolled structure of u yields and the above estimates yields the bound

$$\begin{aligned}
&(1 + \|\mathbb{Y}\|_{\mathcal{Y}^s_T})\|2\Theta_u + 4Y_{\curlyvee}\|_{\mathcal{L}^s_T}\\
&\leq (1 + M)(2\|\tilde{T}_{2\Theta_u+4Y_{\curlyvee}}\Theta\|_{\mathcal{L}^s_T} + 2\|\mathbb{Y}\mathcal{R}^u_{2\Theta_u+4Y_{\curlyvee}}\|_{\mathcal{L}^s_T} + 4\|Y_{\curlyvee}\|_{\mathcal{L}^s_T})\\
&\leq C(1 + M)(\|2\Theta_u + 4Y_{\curlyvee}\|_{\mathcal{L}^r_T}(\|\Theta\|_{C_T(\mathcal{C}^s)} + \|Y\|_{C_T(\mathcal{C}^{s-1})}) + \|\mathbb{Y}\mathcal{R}^u_{2\Theta_u+4Y_{\curlyvee}}\|_{\mathcal{L}^s_T} + 2\|Y_{\curlyvee}\|_{\mathcal{L}^s_T})\\
&\leq C(1 + M^2)(1 + \|u\|_{\mathbb{Y}\mathcal{D}^r_T}),
\end{aligned}$$

from which the required claim follows immdiately. The same argument combined with the multilinearity of our operators also give the second inequality. □

Let us now prove the main theorem.

Proof of Theorem 4.1.3. We will prove the theorem in four steps:

- First, we will demonstrate the existence of locally in time paracontrolled solutions to the (RBE) with $\mathcal{C}^{2s}$ data provided it is driven by signals in $\mathbb{Y}$.
- Second, we will show that the solutions constructed in step one are unique.
- Finally, we will iterate this procedure on small intervals to construct a global solution up to blow-up time.

First Step: Existence of Local Solutions

By density, let $(Z^\epsilon, u_0^\epsilon)$ be a sequence in $\mathcal{L}C(\mathbb{R};\mathcal{D})\times\mathcal{C}^{2s}$ such that

$$\mathbb{Y}^\epsilon = (Y^\epsilon, Y^\epsilon_{\vee}, Y^\epsilon_{\substack{\vee\\\vee}}, \mathring{Y}^\epsilon_{\substack{\vee\\\vee\\\vee}}, Y^\epsilon_{\vee\!\vee}, R(\Theta^\epsilon, Y^\epsilon))$$

are the corresponding enhanced data with

$$\lim_{\epsilon\to 0}\mathbb{Y}^\epsilon_T = \mathbb{Y}_T \text{ in } \mathcal{Y}^s_T \quad\text{and}\quad \lim_{\epsilon\to0} u_0^\epsilon = u_0 \text{ in } \mathcal{C}^{2s},\quad\text{and}$$

$$\max\{\|\mathbb{Y}^\epsilon_T\|_{\mathcal{Y}^s_T}, \|u_0^\epsilon\|_{\mathcal{C}^s}\}\le M_T,\quad (\epsilon,T)\in(0,\infty)\times(0,T^*).$$

Let u^ϵ be the solution to the approximating (RBE):

$$\begin{cases}\mathcal{L}u^\epsilon = \partial_x u^{\epsilon 2} + \partial_x Z^\epsilon\\ u^\epsilon{}_{|t=0} = Y_0^\epsilon + u_0^\epsilon.\end{cases}$$

with paracontrolled derivative $2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}}$ and related structures

$$(\Theta^\epsilon_{u^\epsilon}, \mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}) \;=\; (u^\epsilon - Y^\epsilon - Y^\epsilon_{\vee} - 2Y^\epsilon_{\substack{\vee\\\vee}}, \Theta^\epsilon_{u^\epsilon} - \tilde{T}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}\Theta^\epsilon).$$

In virtue of consideration for short time solutions, fix $t\le 1$ and let $M\overset{\text{def}}{=}M_1$. For every $\epsilon>0$, the remainder term has initial condition

$$\Big(\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}} - Y^\epsilon_{\vee\!\vee} - \mathring{Y}^\epsilon_{\substack{\vee\\\vee\\\vee}}\Big)_{|t=0} = \Theta^\epsilon_{u^\epsilon}{}_{|t=0} - \tilde{T}_{2\Theta^\epsilon_{u^\epsilon}{}_{|t=0}+4Y^\epsilon_{\substack{\vee\\\vee}}{}_{|t=0}}\Theta^\epsilon_{|t=0} = u^\epsilon_{|t=0} = u_0^\epsilon\in\mathcal{C}^{2s}.$$

Thus, applying the Schauder estimate, we can bound

$$\begin{aligned}&\|\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}} - Y_{\vee\!\vee} - \mathring{Y}_{\substack{\vee\\\vee\\\vee}}\|_{\mathcal{L}^{2s}_t}\\ &\qquad\le C(\|e^{t\Delta}u_0\|_{\mathcal{L}^{2s}_t} + \|V_t(\mathcal{L}\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}} - \mathcal{L}Y^\epsilon_{\vee\!\vee} - \mathcal{L}\mathring{Y}^\epsilon_{\substack{\vee\\\vee\\\vee}})\|_{\mathcal{L}^{2s}_t})\\ &\qquad\le C(\|u_0\|_{\mathcal{C}^{2s}} + \|\mathcal{L}\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}} - \mathcal{L}Y^\epsilon_{\vee\!\vee} - \mathcal{L}\mathring{Y}^\epsilon_{\substack{\vee\\\vee\\\vee}}\|_{\mathcal{C}^{2s-2}_t}).\end{aligned}$$

Triangular inequality yields,

$$\|\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}\|_{\mathcal{L}^{2s}} \le \|\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4\dot{Y}^\epsilon_{\substack{\vee\\\vee}}} - Y^\epsilon_{\vee\!\vee} - \mathring{Y}^\epsilon_{\substack{\vee\\\vee\\\vee}}\|_{\mathcal{L}^{2s}_t} + \|Y^\epsilon_{\vee\!\vee} + \mathring{Y}^\epsilon_{\substack{\vee\\\vee\\\vee}}\|_{\mathcal{L}^{2s}_t}.$$

Therefore, by the previous lemma, we readily have

$$\|u^\epsilon\|_{\mathbb{Y}^\epsilon\mathcal{D}^s_t} = \|2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}}\|_{\mathcal{L}^s_t} + \|\mathbb{Y}^\epsilon R^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}\|_{\mathcal{L}^{2s}_t} \le C(1+M^2)(1+\|u^\epsilon\|^2_{\mathbb{Y}^\epsilon\mathcal{D}^r_t}).$$

Applying Lemma 2.5.8, we further get

$$\begin{aligned}\|2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}}\|_{\mathcal{L}^r_t} &\le C(\|(2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}})(0)\|_{\mathcal{C}^r} + t^{(s-r)/2}\|2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}}\|_{\mathcal{L}^s_t}),\quad\text{and}\\ \|\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}\|_{\mathcal{L}^{r+s}_t} &\le C(\|u_0^\epsilon\|_{\mathcal{C}^{r+s}} + t^{(s-r)/2}\|\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}\|_{\mathcal{L}^{2s}}),\end{aligned}$$

then because

$$\|(2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}})(0)\|_{\mathcal{C}^r} \le \|(2\Theta^\epsilon_{u^\epsilon} + 4Y^\epsilon_{\substack{\vee\\\vee}})\|_{\mathcal{L}^s_t}\quad\text{and}\quad \|u_0^\epsilon\|_{\mathcal{C}^{r+s}} \le \|\mathbb{Y}^\epsilon\mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\substack{\vee\\\vee}}}\|_{\mathcal{L}^{2s}},$$

we arrive at

$$\|u^\epsilon\|_{\mathbb{Y}^\epsilon\mathcal{D}^s_t} \le C(1+M^2)(1+t^{s-r}\|u^\epsilon\|^2_{\mathbb{Y}^\epsilon\mathcal{D}^s_t}).$$

By taking t small enough, for all $\epsilon > 0$ there exists $t(\epsilon, M)$, such that

$$\|u^\epsilon\|_{\mathbb{Y}^\epsilon\mathcal{D}^s_t} \le 2C(1+M^2), \quad t \in (0, t(\epsilon, M)].$$

Thus by monotonicity, we get

$$\|u^\epsilon\|_{\mathbb{Y}^\epsilon\mathcal{D}^s_t} \le 2C(1+M^2), \quad t \in (0, t(M)], \quad t(M) \overset{\text{def}}{=} \sup_{\epsilon>0} t(\epsilon, M)$$

uniformly in $\epsilon > 0$. Thus the second part of Lemma 4.1.4 combining with the above estimates for the remainder term yields

$$\begin{aligned}\|2(\Theta^\epsilon_{u^\epsilon} - \Theta^{\epsilon'}_{u^{\epsilon'}} + 4(Y^\epsilon_{\vee} - Y^{\epsilon'}_{\vee}))\|_{\mathcal{L}^s_{t(M)}} + \|\mathbb{Y}^\epsilon \mathcal{R}^{u^\epsilon}_{2\Theta^\epsilon_{u^\epsilon}+4Y^\epsilon_{\vee}} - \mathbb{Y}^{\epsilon'}\mathcal{R}^{u^{\epsilon'}}_{2\Theta^{\epsilon'}_{u^{\epsilon'}}+4Y^{\epsilon'}_{\vee}})\|_{\mathcal{L}^{2s}_{t(M)}} \\ \le C(1+M^2)(\|\mathbb{Y}^\epsilon - \mathbb{Y}^{\epsilon'}\|_{\mathcal{Y}^s_{t(M)}} + \|u^\epsilon_0 - u^{\epsilon'}_0\|_{\mathcal{C}^{2s}}).\end{aligned}$$

for all $t \le t(M)$ and possibility further decreasing $t(M)$ if necessary. Consequentially, this gives rises to the uniform continuity of the local solution map to the (RBE):

$$\tilde{\Phi}_{t(M)} : \tilde{B}_M \to \mathcal{L}^s_{t(M)} \times \mathcal{L}^{2s}_{t(M)}$$

for sufficiently small t which sends each element $(\mathbb{Y}_Z, u_0) \in B_M$ to the corresponding local solution $(u, 2\Theta_u + 4Y_{\vee}) \in \mathcal{L}^s_{t(M)} \times \mathcal{L}^{2s}_{t(M)}$ that solves the (RBE) driven by $\mathbb{Y}_Z$ with initial data u_0, where

$$\tilde{B}_M \overset{\text{def}}{=} \{(\mathbb{Y}_Z, u_0) \mid (Z, u_0) \in \mathcal{LC}(\mathbb{R}; \mathcal{D}) \times \mathcal{C}^{2s}, \max\{\|\mathbb{Y}_Z\|_{\mathcal{Y}^s_1}, \|u_0\|_{\mathcal{C}^{2s}}\} \le M\}.$$

It follows that there exists a unique continuous extension $\Phi_{t(M)}$ of $\tilde{\Phi}_{t(M)}$ onto the closure B_M of $\tilde{B}_M$, which is easily seems to be just the ball centred at 0 with radius M in $\mathcal{Y}^s_1 \times \mathcal{C}^{2s}$. Therefore, for each (RBE)-enhanceable data $\mathbb{Y} \in B_M$, there exists a family $(\mathbb{Y}^\epsilon)_{\epsilon>0}$, for which the corresponding (RBE) driven by $\mathbb{Y}^\epsilon$ with arbitrary initial data (u^ϵ_0) in $\mathcal{C}^{2s}$ is locally solvable with solution u, such that the limit $\lim_{\epsilon\to 0} u^\epsilon = u$ exists in $\mathcal{L}^s_{t(M)} \times \mathcal{L}^{2s}_{t(M)}$ as $(\mathbb{Y}^\epsilon, u^\epsilon_0)$ converges to $(\mathbb{Y}^\epsilon, u_0)$ in $\mathcal{Y}^s_{t(M)} \times \mathcal{C}^{2s}$. By construction, the (RBE) can be properly interpreted on small intervals provided it is driven by signals in $\mathcal{Y}^s_1$. Therefore by passing onto the limit, we see that u solves the (RBE) driven by $\mathbb{Y}$.

Second Step: Uniqueness of Local Solutions

Let u_1 and u_2 be elements in $\mathbb{Y}\mathcal{D}^s_{t(M)}$ with paracontrolled derivatives v_1 and v_2 respectively. Suppose that

$$(\tilde{u}_1, \tilde{u}_2) \overset{\text{def}}{=} (e^{t\Delta}u_0 + V_t(\partial_x u_1^2) + Y,\ e^{t\Delta}u_0 + V_t(\partial_x u_2) + Y), \ (\tilde{u}_1, \tilde{u}_2) \in \mathbb{Y}\mathcal{D}^s_{t(M)} \times \mathbb{Y}\mathcal{D}^s_{t(M)}$$

are a pair of paracontrolled solutions to the (RBE) driven by $\mathbb{Y}$. For all $t \le t(M)$, it is in fact sufficient to estimate

$$\|\mathbb{Y}\mathcal{R}^{\tilde{u}_1}_{2\Theta_{u_1}+4Y_{\vee}} - \mathbb{Y}\mathcal{R}^{\tilde{u}_2}_{2\Theta_{u_2}+4Y_{\vee}}\|_{\mathcal{L}^{2s}_t} \le \|\tilde{u}_1 - \tilde{u}_2\|_{\mathcal{L}^{2s}_t} + 2\|\tilde{T}_{\Theta_{u_1}-\Theta_{u_2}}\Theta\|_{\mathcal{L}^{2s}_t}.$$

On the one hand, by the Schauder estiamte we have

$$\|\tilde{u}_1 - \tilde{u}_2\|_{\mathcal{L}_t^{2s}} \leq C\|\partial_x(u_1^2 - u_2^2)\|_{C_t(\mathcal{C}^{2s-2})}.$$

Since

$$\begin{aligned}
&\partial_x(u_1^2 - u_2^2) \\
&\quad = 2\partial_x((\Theta_{u_1} - \Theta_{u_2})Y) + 2\partial_x((\Theta_{u_1} - \Theta_{u_2})Y_{\vee}) + \partial_x(\Theta_{u_1}^2 - \Theta_{u_2}^2) + 4\partial_x((\Theta_{u_1} - \Theta_{u_2})Y_{\curlyvee}).
\end{aligned}$$

By bounding individual components and the fact that

$$\max\{\|\Theta_{u_1}\|_{\mathcal{L}_t^s}, \|\Theta_{u_2}\|_{\mathcal{L}_t^s}\} \leq C(1 + M^2), \quad t \leq t(M),$$

we get

$$\begin{aligned}
\|\tilde{u}_1 - \tilde{u}_2\|_{\mathcal{L}_t^{2s}} &\leq C(1 + M^2)\|\Theta_{u_1} - \Theta_{u_2}\|_{C_t(\mathcal{C}^s)} \\
&\leq C(1 + M^2)(\|v_1 - v_2\|_{\mathcal{L}_t^s} + \|\mathbb{Y}\mathcal{R}_{v_1}^{u_1} - \mathbb{Y}\mathcal{R}_{v_1}^{u_1}\|_{\mathcal{L}_t^{2s}})
\end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned}
\|\tilde{T}_{\Theta_{u_1} - \Theta_{u_2}}\Theta\|_{\mathcal{L}_t^{2s}} \leq \|\mathcal{L}\tilde{T}_{\Theta_{u_1} - \Theta_{u_2}}\Theta\|_{C_t(\mathcal{C}^{2s-2})} &\leq CM\|\Theta_{u_1} - \Theta_{u_2}\|_{\mathcal{L}_T^{2s}} \\
&\leq CM(\|\mathcal{L}\tilde{T}_{v_1 - v_2}\Theta\|_{C_t(\mathcal{C}^{2s-2})} + \|\mathbb{Y}\mathcal{R}_{v_1}^{u_1} - \mathbb{Y}\mathcal{R}_{v_2}^{u_2}\|_{\mathcal{L}_t^s}) \\
&\leq (1 + C)M^2(\|v_1 - v_2\|_{\mathcal{L}_t^s} + \|\mathbb{Y}\mathcal{R}_{v_1}^{u_1} - \mathbb{Y}\mathcal{R}_{v_2}^{u_2}\|_{\mathcal{L}_t^{2s}}).
\end{aligned}$$

Therefore, we have

$$\|\mathbb{Y}\mathcal{R}_{2\Theta_{u_1} + 4Y_{\curlyvee}}^{\tilde{u}_1} - \mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_2}\|_{\mathcal{L}_t^{2s}} \leq C(1 + M^2)\|(u, v)\|_{\mathbb{Y}\mathcal{D}_t^s}.$$

Now by Lemma 4.1.4 and the Schauder estimate, we deduce

$$\begin{aligned}
\|\Theta_{\tilde{u}_1} - \Theta_{\tilde{u}_2}\|_{\mathcal{L}_t^s} &\leq t^{(s-r)/2}CM^2(\|\Theta_{u_1} - \Theta_{u_2}\|_{\mathcal{L}_t^s} + \|\mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_1} - \mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_2}\|_{\mathcal{L}^{2s}}), \\
\|\mathbb{Y}\mathcal{R}_{2\Theta_{u_1} + 4Y_{\curlyvee}}^{\tilde{u}_1} - \mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_2}\|_{\mathcal{L}_t^{2s}} &\leq C\|\mathcal{L}\mathbb{Y}\mathcal{R}_{2\Theta_{u_1} + 4Y_{\curlyvee}}^{\tilde{u}_1} - \mathcal{L}\mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_2}\|_{C_t(\mathcal{C}^{2s-2})} \\
&\leq t^{(s-r)/2}CM^2(\|\Theta_{u_1} - \Theta_{u_2}\|_{\mathcal{L}_t^s} + \|\mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_1} - \mathbb{Y}\mathcal{R}_{2\Theta_{u_2} + 4Y_{\curlyvee}}^{\tilde{u}_2}\|_{\mathcal{L}^{2s}}).
\end{aligned}$$

Putting everything together finally yields

$$\|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{Y}\mathcal{D}_t^s} \leq t^{(s-r)/2}C(1 + M^2)\|(u_1 - u_2, v_1 - v_2)\|_{\mathbb{Y}\mathcal{D}_t^s}.$$

Taking $t \leq t^*(M)$ for sufficiently small $t^*(M)$ thus gives the required uniqueness of paracontrolled solutions. In particular, we reduce $t(M)$ to $t^*(M)$ if necessary.

Final Step: Iteration on Unit Time Intervals

It remains to iterate this argument in order to obtain solutions up to the blow up time T^*. Let $T, r > 0$ and $[0, T]$ and $[T, T+1]$ be two intervals. Without loss of generality, assume there exists a unique paracontrolled solution u which solves the (RBE) driven by $\mathbb{Y}$ on $[0, T]$. Consider the translated (RBE) on $[T, T+1]$:

$$\begin{cases} \mathcal{L}\tilde{u} = \partial_x \tilde{u}^2 + \partial_x Z \\ \tilde{u}_{|t=0} = \tilde{u}_T. \end{cases}$$

Moreover, we let $\tilde{\mathbb{Y}}_t \overset{\text{def}}{=} \mathbb{Y}_{T+t}$. Since the initial condition $\tilde{u}_T$ is not in $\mathcal{C}^{2s}$ but rather $\mathcal{C}^{s-1}$. This is not the same situation as before. Nevertheless, we only needed a smooth initial condition to obtain an initial condition of regularity $2s$ for the remainder, and now we have

$$\tilde{\mathbb{Y}}\mathcal{R}^{\tilde{u}}_{2\Theta_{\tilde{u}}+4\tilde{Y}_{\vee\!\vee}}\Big|_{t=0} = u(T) - Y(T) - Y_{\vee}(T) - 2Y_{\vee\!\vee}(T) - (T_{2\Theta_u+4Y_{\vee\!\vee}}\Theta)(T).$$

By Lemma 2.5.2,

$$\|(\tilde{T}_{2\Theta_u+4Y_{\vee\!\vee}}\Theta - T_{2\Theta_u+4Y_{\vee\!\vee}}\Theta)(T)\|_{\mathcal{C}^{2s}} \leq C_s\|(2\Theta_u + 4Y_{\vee\!\vee}\Theta)(T)\|_{\mathcal{C}^s}\|\Theta(T)\|_{\mathcal{C}^s}$$

and so the paracontrolled structure of u on $[0,T]$ shows that the initial condition satisfies $\tilde{\mathbb{Y}}\mathcal{R}^{\tilde{u}}_{2\Theta_{\tilde{u}}+4\tilde{Y}_{\vee\!\vee}}\big|_{t=0} \in \mathcal{C}^{2s}$. By was proved, we can find $t(\tilde{M})$ small enough, depending on a positive constant $\tilde{M}$, and a unique paracontrolled solution $\tilde{u}$ which solves the translated (RBE) on $[T, T+t(\tilde{M})]$. We then extend the paracontrolled structure $(u, 2\Theta_u + 4Y_{\vee\!\vee})$ to the interval $[0, T+t(\tilde{M})]$ by the obvious formulation

$$u(t) \overset{\text{def}}{=} \tilde{u}(t-T),\ (2\Theta_u + 4Y_{\vee\!\vee})(t) \overset{\text{def}}{=} (2\Theta_u + 4Y_{\vee\!\vee})(t-T),\ t \in [T, T+t(\tilde{M})].$$

Consequentially, the remainder is extended as well. □

4.2 From the (RBE) to the (KPZ)

It remains to transfer what was proved for the (RBE) to that of the (KPZ). Before we do this, it is however necessary to note that one doesn't solve directly the (KPZ), but instead consider solutions over a class of *renormalised equations*

4.2.1 Renormalisations

Consider the (KPZ) equation (4.1) in one-dimension with zero initial data. As in the case of the (RBE), one could expect to expand u into a linear combination of finer stochastic components

$$h = W + \tilde{W}_{\vee} + 2W_{\vee\!\vee} + 4\tilde{W}_{\vee\!\vee\!\vee} + \tilde{W}_{\vee\vee} + \mathcal{R}h \tag{4.5}$$

with the iteration properties that

$$\mathcal{L}W = Z,\ \ \mathcal{L}\tilde{W}_{\vee} = (\partial_x W)^2,\ \ \mathcal{L}W_{\vee\!\vee} = (\partial_x W)(\partial_x \tilde{W}_{\vee}),$$
$$\mathcal{L}\tilde{W}_{\vee\!\vee\!\vee} = (\partial_x \tilde{W}_{\vee\!\vee})(\partial_x W),\ \ \mathcal{L}\tilde{W}_{\vee\vee} = (\partial_x \tilde{W}_{\vee})(\partial_x \tilde{W}_{\vee}),$$

and so on, such that the equation can be made sense of via the Littlewood-Paley theory provided each of the above elements in the decomposition satisfies some a priori regularity assumptions. We have shown, in the case of the (RDE) and (RBE), that if one assumes such construction can be done via smooth sequences of regularised stochastic data X^ϵ and Z^ϵ, such that $\lim_{\epsilon\to 0} X^\epsilon = X$ and $\lim_{\epsilon\to 0} Z^\epsilon = Z$, then the solutions to these regularised stochastic equations converge to the unique solutions of the actual stochastic equations.

On the other hand, in the stochastic settings, the (KPZ) exhibits an unfavourable phenomenon that no such approximate solutions can be shown to converge. For no smooth sequences $(Z^\epsilon)_{\epsilon>0}$ does it happen that the solutions h^ϵ to the (KPZ) with smooth noises

Z^ϵ converges to any limit. In the case of the finer decomposition (4.5), it is precisely that fact that $\tilde{W}_{\vee}$, $\tilde{W}_{\vee_{\vee}}$ and $\tilde{W}_{\vee\vee}$ fail to converge. Such a problem is solved if one considers a sequence $(c^\epsilon_{\vee}, c^\epsilon_{\vee_{\vee}})_{\epsilon>0}$ of constants which diverges as ϵ goes to zero in such a way that $\lim_{\epsilon\to 0}(\tilde{W}^\epsilon_{\vee} - tc^\epsilon_{\vee})$, $\lim_{\epsilon\to 0}(\tilde{W}_{\vee_{\vee}} + tc_{\vee\vee}4^{-1})$ and $\lim_{\epsilon\to 0}(\tilde{W}^\epsilon_{\vee\vee} - tc^\epsilon_{\vee\vee})$ exist as finite limits. One could show that the construction of such constants are possible. In the physical literature, it is stated that the equation remains to be a good description of physical phenomenons even after the applications of such renormalisations.

Notice that we have chosen the constants $(c_{\vee_{\vee}}, c_{\vee\vee})$ such that $c_{\vee\vee}4^{-1}$ and $c_{\vee\vee}$ cancel exactly on the level of the equation. In fact, with these particular choices of renormalisation constants, we instead solve the renormalised equation $(KPZ)_r$:

$$\begin{cases} \mathcal{L}h = (\partial_x h)^{\diamond 2} + Z, \quad (\partial_x h)^{\diamond 2} \overset{\text{def}}{=} (\partial_x h)^2 - c_{\vee_{\vee}} \\ h_{|t=0} = 0. \end{cases}$$

This motivates the following definitions.

Definition 4.2.1. Let s be in $(1/3, 1/2)$. The space of (KPZ) enhanceable elements $\mathcal{W}^s$ consists of all those distributions $\mathbb{W}(c_{\vee}, c_{\vee_{\vee}})$ in the closure for the image of the map

$$\mathcal{L}C^{s/2}_{loc}(\mathbb{R}; \mathcal{D}(\mathbb{T})) \to \mathcal{L}^s \times \mathcal{L}^{2s} \times \mathcal{L}^{s+1} \times \mathcal{L}^{2s+1} \times \mathcal{L}^{2s+1} \times C(\mathcal{C}^{2s-1})$$

defined by

$$Z \longmapsto (W, W_{\vee}, W_{\vee_{\vee}}, \mathring{W}_{\vee_{\vee}}, W_{\vee\vee}, (\partial_x P, \partial_x Y))$$

associated with a pair of constants $(c_{\vee}, c_{\vee_{\vee}})$, and the elements satisfying

$$\mathcal{L}W = Z, \quad \mathcal{L}W_{\vee} = (\partial_x W)^2 - c_{\vee}, \quad \mathcal{L}W_{\vee_{\vee}} = (\partial_x W)(\partial_x W_{\vee}),$$

$$\mathcal{L}\mathring{W}_{\vee_{\vee}} = R(\partial_x W_{\vee_{\vee}}, \partial_x W) + c_{\vee\vee}4^{-1}, \quad \mathcal{L}W_{\vee\vee} = (\partial_x W_{\vee})(\partial_x W_{\vee}) - c_{\vee\vee}, \quad \mathcal{L}\mathit{\Gamma} = \partial_x W,$$

and initial conditions

$$(W_{\vee}, W_{\vee_{\vee}}, \mathring{W}_{\vee_{\vee}}, W_{\vee\vee}, \mathit{\Gamma})_{|t=0} = 0 \quad \text{and} \quad W_{|t=0} = \int_{-\infty}^0 e^{-t\Delta}(\mathbb{P}Z)ds,$$

pointwise in $\mathcal{L}C^{s/2}_{loc}(\mathbb{R}; \mathcal{D}(\mathbb{T}))$, where $\mathbb{P}$ denotes the projections on the non-zero spatial Fourier modes. The initial condition of W is also denoted by W_0. In practice we committed the dependency on $(c_{\vee}, c_{\vee_{\vee}})$ in the notation and simply write $\mathbb{W}$. For every $T > 0$, the normed space $(\mathcal{W}^s_T, \|\cdot\|_{\mathcal{W}^s_T})$ is

$$\mathcal{W}^s_T \overset{\text{def}}{=} \mathcal{W}^s{}_{|[0,T]} \quad \mathbb{W}_T \overset{\text{def}}{=} \mathbb{W}_{|[0,T]} \text{ for every } \mathbb{W} \in \mathcal{W}^s, \quad \text{and}$$

$$\begin{aligned} \|\mathbb{W}_T\|_{\mathcal{W}^s_T} \overset{\text{def}}{=} \|W\|_{C_T(\mathcal{C}^s)} + \|W_{\vee}\|_{C_T(\mathcal{C}^{2s})} + \|W_{\vee_{\vee}}\|_{\mathcal{L}^{s+1}_T} \\ + \|\mathring{W}_{\vee_{\vee}}\|_{\mathcal{L}^{2s+1}_T} + \|W_{\vee\vee}\|_{\mathcal{L}^{2s+1}_T} + \|R(\partial_x \mathit{\Gamma}, \partial_x Y)\|_{C_T(\mathcal{C}^{2s-1})}. \end{aligned}$$

Remark 4.2.2. By definition, for every $\mathbb{W} \in \mathcal{W}^s$ we can find an element $\mathbb{Y} \in \mathcal{Y}^s$, defined by taking the spatial derivatives of the elements $(W, W_{\vee}, W_{\vee_{\vee}}, \mathring{W}_{\vee_{\vee}}, W_{\vee\vee}, \mathit{\Gamma})$. In this case, with a slight abuse of notation we will write $\mathbb{Y} = \partial_x \mathbb{W}$.

We also have the corresponding class of paracontrolled distributions.

Definition 4.2.3. Let $s \in (1/2, 1/2)$ and $\mathbb{W}$ be in $\mathcal{W}^s$. A pair of distributions $(h, v) \in C(\mathcal{C}^s) \times \mathcal{L}^r$ is paracontrolled by $\mathbb{W}$ if there exists $\mathbb{W}\mathcal{R}_v^h \in \mathcal{L}^{s+r+1}$ such that

$$h = W + W_{\vee} + 2W_{\vee\!\!\vee} + \tilde{T}_v \mathit{\Gamma} + \mathbb{W}\mathcal{R}_v^h.$$

We say that v is the derivative of h paracontrolled by $\mathbb{W}$. The space $\mathbb{W}\mathcal{D}^r$ consists of all such distributions

$$\mathbb{W}\mathcal{D}^r \stackrel{\text{def}}{=} \{(h, v) \in C(\mathcal{C}^s) \times \mathcal{L}^r \mid \mathbb{W}\mathcal{R}_v^h \stackrel{\text{def}}{=} h - W - W_{\vee} - 2W_{\vee\!\!\vee} - \tilde{T}_v \mathit{\Gamma} \in \mathcal{L}^{s+r+1}\}.$$

For every $T > 0$, the normed space $(\mathbb{W}\mathcal{D}_T^s, \|\cdot\|_{\mathbb{W}\mathcal{D}_T^r})$ is

$$\mathbb{W}\mathcal{D}_T^r \stackrel{\text{def}}{=} \mathbb{W}\mathcal{D}^r{}_{|[0,T]} \quad \text{and}$$

$$\|(h, v)\|_{\mathbb{W}\mathcal{D}_T^r} \stackrel{\text{def}}{=} \|v\|_{\mathcal{L}_T^r} + \|\mathbb{W}\mathcal{R}_v^h\|_{\mathcal{L}_T^{s+r+1}}.$$

It is clear that $(\mathbb{W}\mathcal{D}_T^r, \|\cdot\|_{\mathbb{W}\mathcal{D}_T^r})$ is complete. We call h a *paracontrolled solution to the* $(KPZ)_r$ if $(h, 2\partial_x \mathit{\Gamma} + 4\partial_x W_{\vee\!\!\vee})$ is in $\mathbb{W}\mathcal{D}^s$. In that case we also write $\|h\|_{\mathbb{W}\mathcal{D}_T^s} = \|(h, 2\partial_x \mathit{\Gamma} + 4\partial_x W_{\vee\!\!\vee})\|_{\mathbb{W}\mathcal{D}_T^s}$.

4.2.2 Well-posedness of the (KPZ) with $\mathcal{C}^{2s+1}$ Data

Using the well-posedness result for the (RBE) with $\mathcal{C}^{2s}$ data. It is not difficult to establish analogous statement for the (KPZ) with $\mathcal{C}^{2s+1}$ data. Indeed, suppose a prior that h is a solution to the $(KPZ)_r$ driven by $\mathbb{W}$, then $\partial_x h$ solves the (RBE) driven by $\partial_x \mathbb{W}$. Hence, we expect h to satisfy the more general equation

$$\mathcal{L}h = u^{\diamond 2} + Z, \quad h_{|t=0} = Y(0) + h_0 \tag{4.6}$$

where $u \in (\partial_x \mathbb{W})\mathcal{D}^s$ is the unique paracontrolled solution to the corresponding (RBE).

Theorem 4.2.4. *Let $(\mathbb{W}, h_0)$ be in $\mathcal{W}^s \times \mathcal{C}^{2s+1}$. Then there exists a unique paracontrolled solution $(h, v) \in \mathbb{W}\mathcal{D}^s$ to the (RBE) with initial condition $W(0) + h_0$ for all $T < T^*$, where*

$$T^* \stackrel{def}{=} \inf_{\substack{T \in \mathbb{R}^+ \\ \|h\|_{\mathbb{W}\mathcal{D}_T^s = \infty}}} T.$$

Proof. Let h be given as an arbitrary tempered distribution and consider the equation (4.6) tested against all $u \in (\partial_x \mathbb{W})\mathcal{D}^s$ such that u is a paracontrolled solution to the (RBE) driven by $\partial_x \mathbb{W}$. Then u admits the paracontrolled structure

$$\begin{cases} u = \partial_x W + \partial_x W_{\vee} + 2\partial_x W_{\vee\!\!\vee} + \tilde{T}_{2\Theta_u + 4\partial_x W_{\vee\!\!\vee}} \Theta + (\partial_x \mathbb{W})\mathcal{R}^u_{2\Theta_u + 4\partial_x W}, \\ \Theta_u = \partial_x h - \partial_x W - \partial_x W_{\vee} - 2\partial_x W_{\vee\!\!\vee}, \ (2\Theta_u + 4\partial_x W_{\vee\!\!\vee}, \ (\partial_x \mathbb{W})\mathcal{R}^u_{2\Theta_u + 4\partial_x W_{\vee\!\!\vee}}) \in \mathcal{L}^s \times \mathcal{L}^{2s}. \end{cases} \tag{4.7}$$

Set

$$\begin{cases} \tilde{h} \stackrel{\text{def}}{=} W + W_{\vee} + 2W_{\vee\!\!\vee} + \tilde{T}_{2\Theta_u + 4\partial_x W} \mathit{\Gamma} + \mathbb{W}\mathcal{R}^u_{2\Theta_u + 4\partial_x W_{\vee\!\!\vee}}, \ \text{and} \\ g \stackrel{\text{def}}{=} (\partial_x \mathbb{W})\mathcal{R}^u_{2\Theta_u + 4\partial_x W_{\vee\!\!\vee}} - \tilde{T}_{2\partial_x \Theta_u + 4\partial_x^2 W_{\vee\!\!\vee}} \mathit{\Gamma} - \partial_x(\mathbb{W}\mathcal{R}^u_{2\Theta_u + 4\partial_x W}), \end{cases}$$

where we have

$$(\tilde{T}_{2\partial_x\Theta_u+4\partial_x^2 W_{\curlyvee}}\Gamma, \partial_x(\mathbb{W}\mathcal{R}^u_{2\Theta_u+4\partial_x W}), (\partial_x\mathbb{W})\mathcal{R}^u_{2\Theta_u+4\partial_x W}) \in C(\mathcal{C}^{2s}) \times \mathcal{L}^{2s} \times \mathcal{L}^{2s}$$
$$\implies g \in C(\mathcal{C}^{2s}).$$

Then taking derivative yields

$$\partial_x \tilde{h} = \partial_x W + \partial_x W_{\vee} + 2\partial_x W_{\curlyvee} + \tilde{T}_{2\partial_x\Theta_u+4\partial_x^2 W}\Gamma + (\partial_x\mathbb{W})\mathcal{R}^u_{2\Theta_u+4\partial_x W_{\curlyvee}}$$
$$- (\partial_x\mathbb{W})\mathcal{R}^u_{2\Theta_u+4\partial_x W_{\curlyvee}} + \tilde{T}_{2\Theta_u+4\partial_x W}\Theta + \partial_x(\mathbb{W}\mathcal{R}^u_{2\Theta_u+4\partial_x W}) = u - g,$$

or $u = \partial_x\tilde{h} + g$ for $(\tilde{h}, g) \in \mathbb{W}\mathcal{D}^s \times C(\mathcal{C}^{2s})$. Now, expanding h into the decomposition

$$h = W + W_{\vee} + 2W_{\curlyvee} + \Gamma_h$$

and examine the finer structure of the $(KPZ)_r$, we obtain

$$\mathcal{L}\Gamma_h = \mathcal{L}(h - W - W_{\vee} - 2W_{\curlyvee}) = u^{\diamond 2} - (\partial_x W)^2 - c_{\vee} - 2(\partial_x W)(\partial_x W_{\vee}).$$

Noting that $u^{\diamond 2} = (\partial_x\tilde{h})^2 - c_{\vee} + 2g(\partial_x\tilde{h}) + g^2$, we have

$$u^{\diamond 2} - (\partial_x W)^2 + c_{\vee} - 2(\partial_x W)(\partial_x W_{\vee}) =$$
$$(\partial_x\Gamma_{\tilde{h}})^2 + (\partial_x W_{\vee})^2 + 4(\partial_x W_{\curlyvee})^2 + 2(\partial_x\Gamma_{\tilde{h}})(\partial_x W) + 2(\partial_x\Gamma_{\tilde{h}})(\partial_x W_{\vee})$$
$$+ 3(\partial_x\Gamma_{\tilde{h}})(\partial_x W_{\curlyvee}) + 3(\partial_x W)(\partial_x W_{\curlyvee}) + 4(\partial_x W_{\vee})(\partial_x W_{\curlyvee})$$
$$+ 2g(\partial_x\Gamma_{\tilde{h}}) + 2g(\partial_x W) + g(\partial_x W_{\vee}) + 2g(\partial_x W_{\curlyvee}) + g^2.$$

We would like to prove that the right hand side of this expression defines an element in the image $\mathcal{L}(\mathcal{L}^{s+1})$. Nevertheless, the only terms which requires different arguments are $(\partial_x W_{\vee})^2$, $(\partial_x\Gamma_{\tilde{h}})(\partial_x W)$ and $(\partial_x W)(\partial_x W_{\curlyvee})$. But for these terms, we can apply the structure of the enhanced data $\mathbb{W} \in \mathcal{W}^s$ to see that

$$(\partial_x W_{\vee})^2 = \mathcal{L}(W_{\vee\vee} + tc_{\vee\vee}), \quad \text{and} \quad R(\partial_x W_{\curlyvee}, \partial_x W) = \mathcal{L}(\mathring{W}_{\substack{\curlyvee \\ \vee}} - tc_{\vee\vee}4^{-1}),$$

and from here we have $(\partial_x W_{\vee}, (\partial_x W)(\partial_x W_{\curlyvee})) \in \mathcal{L}(\mathcal{L}^{s+1}) \times \mathcal{L}(\mathcal{L}^{s+1})$. On the other hand, writing out

$$(\partial_x\Gamma_{\tilde{h}})(\partial_x W) + g(\partial_x W) = (\tilde{T}_{2\Theta_u+4\partial_x W_{\curlyvee}}\Theta + (\partial_x\mathbb{W})\mathcal{R}^u_{2\Theta_u+4\partial_x W_{\curlyvee}})(\partial_x W)$$

and making an application of Proposition 3.2.10, we have

$$\|(\partial_x\Gamma_{\tilde{h}} + g)(\partial_x W) - T_{\partial_x\Gamma_{\tilde{h}}+g}(\partial_x W)\|_{C_T(\mathcal{C}^{s-1})}$$
$$\leq C(\|2\Theta_u + 4\partial_x W_{\curlyvee}\|_{C_T(\mathcal{C}^s)}\|\Theta\|_{C_T(\mathcal{C}^s)} + \|(\partial_x\mathbb{W})\mathcal{R}^u_{2\Theta_u+4\partial_x W_{\curlyvee}}\|_{C_T(\mathcal{C}^s)})$$
$$\times (\|\partial_x W\|_{C_T(\mathcal{C}^{s-1})}) + \|2\Theta_u + 4\partial_x W_{\curlyvee}\|_{C_T(\mathcal{C}^s)}\|R(\Theta, \partial_x W)\|_{C_T(\mathcal{C}^{2s-1})}, \quad T > 0.$$

Thus, combining with the obvious fact that $\tilde{T}_{\partial_x\Gamma_{\tilde{h}}+g}(\partial_x W)$ is in $C(\mathcal{C}^{s-1})$ we can conclude that the additive term $(\partial_x\Gamma_{\tilde{h}})(\partial_x W)$ is also in $C(\mathcal{C}^{s+1})$. Estimations for the norms of the remaining terms can be easily calculated using the Littlewood-Paley theory and is thus omitted. In particular, we have that Γ_h defines an element of $\mathcal{L}^{s+1}$.

Using this fact, it is then enough to note that

$$\mathcal{L}\mathbb{W}\mathcal{R}^u_{2\Theta_u+4\partial_x W_{\vee}} = \mathcal{L}\Gamma_h - (\mathcal{L}\tilde{T}_{2\Theta_u+4\partial_x W}\Gamma + \tilde{T}_{2\Theta_u+4\partial_x W_{\vee}}\mathcal{L}\Gamma) - \tilde{T}_{2\Theta_u+4\partial_x W_{\vee}}\mathcal{L}\Gamma.$$

Then applying again Lemma 2.5.2 gives that $\mathbb{W}\mathcal{R}^u_{2\Theta_u+4\partial_x W_{\vee}}$ is in $\mathcal{L}^{2s+1}$ to see that h defines an element of $\mathbb{W}\mathcal{D}^s$ with paracontrolled derivative $2\Theta_u + 4\partial_x W_{\vee}$.

A priori, $\partial_x h$ solves the equation

$$\mathcal{L}(\partial_x h) = \partial_x u^2 + \partial_x Z, \quad \partial_x h_{|t=0} = \partial_x W(0) + \partial_x h_0.$$

But also u is the unique paracontrolled solution to the (RBE)

$$\mathcal{L}u = \partial_x u^2 + \partial_x Z, \quad u_{|t=0} = u_0.$$

Thus if we require that $u_0 = \partial_x W(0) + \partial_x h_0$, then

$$\mathcal{L}(\partial_x h - u) = 0, \quad \partial_x h_{|t=0} - u_{|t=0} = 0$$

so that by the Schauder estimate $\partial_x h = u$. Therefore, if h_1 and h_2 are two solutions to the (KPZ), then they must take the form

$$h_1 = u + C_1(t) \quad \text{and} \quad h_2 = u + C_2(t)$$

for some C_1, C_2 depending on t only. But

$$\begin{aligned}\partial_t(C_1(t) - C_2(t)) &= \partial_t h_1 - \partial_t h_2 \\ &= (\partial_x^2 h_1 - \partial_x^2 h_2) + ((\partial_x h_1)^2 - (\partial_x h_2)^2) \\ &= (\partial_x u - \partial_x u) + (u^2 - u^2) = 0.\end{aligned}$$

So we actually have $C_1(t) - C_2(t) = C$ is a constant. The initial condition now determines that C must be zero. Therefore h is a unique solution to the (KPZ), which exists up to blow-up time. □

4.3 References and Remarks

The (KPZ) had been a long-standing problem in the field of stochastic partial differential equations before the one-dimensional case was first solved by M. Hairer in his remarkable paper [9]. Many substantial contributions have appeared ever since, including the method of paracontrolled distributions which we introduced here, as well as other probabilistic results obtained independently by for instance I. Corwin. The construction in this chapter is mostly due to M. Gubinelli and can be found in [6].

It might seems that many assumptions we have made throughout this chapter have not really been properly put in uses. Indeed, out analysis of the (KPZ) has not concluded yet. It remains to construct explicitly those regularisations $\mathbb{Y}$ such that the corresponding convergence criteria is satisfied almost surely. The violation caused by the white noise on $\mathbb{R}$ is actually not too serious, since everything in this chapter works equally well on the space as soon as we are willing to assume $Z \in C(\mathcal{C}^{-1/2-})$. Evidently, our methods have been recently extended to $\mathbb{R}$ by N. Perkowski and T.-C. Rosati, there the arguments are analogous except one must adjust to a space of weighted tempered distributions, which we chose not to present here. See [8] for more details.

The terrible choice of notations introduced in Definition 4.1.1 and 4.2.1 are not without reasons. These were the original notations used by M. Hairer, where the author considered algebraic method, in which case the symbols $\vee, \checkmark\!\!\vee, \ldots$ should be interpreted as binary trees, thus the notations are actually natural. However, due to the extreme success of that work, such notations have become a standard in the field. In fact, for the (RBE) one could express the solution u as an infinite sum of terms labelled by binary trees:

$$u = \sum_{\tau \in \mathcal{T}} c(\tau) Y_\tau, \quad \text{with} \quad Y_\tau \stackrel{\text{def}}{=} \mathcal{B}(\tau_1, \tau_2).$$

Where $\mathcal{T}$ is the space of binary trees, τ_1 and τ_2 are chosen such that $\tau_1 \cdot \tau_2 = \tau$ and $c(\tau)$ is the combinatorial factors counting the number of planner trees which are isomorphic as graphs to τ. If $d(\tau)$ denotes the degree of τ, then one obtains in general

$$u = \sum_{\tau \in \mathcal{T}, d(\tau) < n} c(\tau) Y_\tau + \mathcal{R}^{(n)} u,$$

where the remainder satisfies the recursive relation

$$R^{(n)} u = \sum_{d(\tau_1) < n, d(\tau_2) < n. d(\tau_1 \cdot \tau_2) \geq n} c(\tau_1) c(\tau_2) Y_{\tau_1 \cdot \tau_2} + \sum_{d(\tau) < n} c(\tau) \mathcal{B}(Y_\tau, \mathcal{R}^{(n)} u) + \mathcal{B}(\mathcal{R}^{(n)} u, \mathcal{R}^{(n)} u).$$

In our case, we had

$$c(\cdot) = 1, \; c(\vee) = 1, \; c(\checkmark\!\!\vee) = 2, \; c(\checkmark\!\!\checkmark\!\!\vee) = 4, \; c(\vee\!\!\vee\!\!\vee) = 1$$

with $Y_\cdot = Y$ and $n = 3$.

Bibliography

[1] H. Bahouri, J.-Y. Chemin and R.Danchin: Fourier Analysis and Nonlinear Partial Differential Equations, *Grundlehren der mathematischen Wissenschaften*, **343**, Springer (2011)

[2] M. Gubinelli, P. Imkeller and N. Perkowski: Paracontrolled Distributions And Singular Pdes, *Forum of Mathematics, Pi*, **3**, (2015).

[3] M. Gubinelli, P. Imkeller and N. Perkowski: A Fourier analytic approach to pathwise stochastic integration, *Electronic Journal of Probability*, **21**(0), (2016).

[4] M. Gubinelli and N. Perkowski: KPZ Reloaded, *Communications in Mathematical Physics*, **349**(1), (2016), pages 165–269.

[5] M. Hairer: Solving the KPZ equation, *Annals of Mathematics*, **178**(2), (2013), pages 559–664.

[6] M. Gubinelli and N. Perkowski: Lectures on singular stochastic PDEs, *Preprint.*

[7] T.-J. Lyons, M, Caruana, T. Lévy: *Differential Equations Driven by Rough Paths*, *École dÉté de Probabilités de Saint-Flour XXXIV - 2004. Springer, Berlin*, (2007).

[8] N. Perkowski and T.-C. Rosati: The KPZ Equation on the Real Line, *Preprint.*

[9] D.J. Prömel and M, Trabs: Rough differential equations driven by signals in Besov spaces, *Journal of Differential Equations*, **260**(6), (2016), pages 5202–5249.